A Constrained Representation Theorem for Interval Type-2 Fuzzy Sets Using Convex and Normal Embedded Type-1 Fuzzy Sets, and Its Application to Centroid Computation

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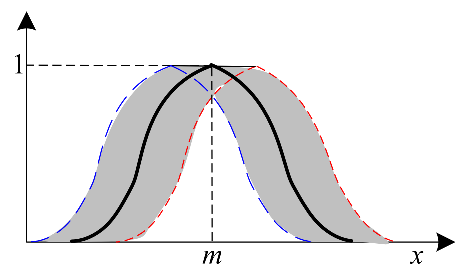
***Abstract*—The Representation Theorem for interval type-2 fuzzy sets (IT2 FSs), proposed by Mendel and John, states that an IT2 FS is a combination of all its embedded type-1 (T1) FSs, which can be non-convex and/or sub-normal. These non-convex and/or sub-normal embedded T1 FSs are included in developing many results for IT2 FSs, including the uncertainty measures and linguistic weighted average. However, almost all applications of fuzzy logic systems use only convex and normal FSs. In this paper, we propose a Constrained Representation Theorem for IT2 FSs using only convex and normal embedded T1 FSs. We also apply it to computing the centroid of IT2 FSs, the most important property and uncertainty measure of IT2 FSs. This Constrained Representation Theorem can serve as the basis for developing many theoretical results, as the Mendel-John Representation Theorem has done.**

***Index Terms*—Representation Theorem, Constrained Representation Theorem, interval type-2 fuzzy set, centroid, constrained centroid**

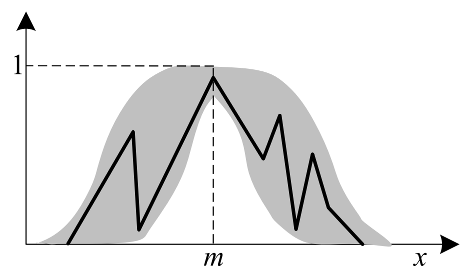
I. Introduction

Interval type-2 fuzzy sets (IT2 FSs) and systems [9], [27] have been gaining popularity rapidly in the last decade. The Mendel-John Representation Theorem [12] for IT2 FSs has played an important role. It states that the footprint of uncertainty (FOU) of an IT2 FS is the union of all its embedded type-1 (T1) FSs, including those that are non-convex and/or sub-normal. This Representation Theorem implies that all these embedded T1 FSs should be considered in deriving new theoretical results for IT2 FSs [16]. In fact, it has been used in defining uncertainty measures [21], [23], similarity measures [17], [18], [23], [26], subsethood measures [17], [26], the linguistic weighted average [13], [20], [22], [25], etc.

However, it must be noted that almost all applications of fuzzy logic systems use only convex and normal FSs. So, using non-convex and/or sub-normal embedded T1 FSs in IT2 FSs and systems seems controversial. Some researchers have noticed this problem and proposed to use constrained embedded T1 FSs. Garibaldi [4] pointed out that *"conventional type-2 fuzzy sets also suffer from the problem that they contain embedded sets that do not correspond to meaningful concepts."* He gave two examples, as shown in Fig. 1, where the FOU is obtained by blurring the mean of a Gaussian T1 FS. Consequently, the three Gaussian T1 FSs in Fig. 1(a) are *"meaningful"* embedded T1 FSs, whereas the T1 FS in Fig. 1(b) is *"technically possible but meaningless."* He then proposed *"constrained type-2 fuzzy sets,"* where only meaningful T1 FSs are considered as embedded T1 FSs. Aisbett et al. [1] also proposed the concept of *"constrained embedded membership function (MF),"* which is essentially the same as Garibaldi's idea, i.e., all constrained embedded MFs should assume similar meaningful functional form, and generally they are convex and normal.



(a)



(b)

Fig. 1. Illustration of (a) "meaningful" and (b) "meaningless" embedded T1 FSs.

The above two ideas are very useful when we know exactly how an IT2 FS is constructed. For example, if we know an IT2 FS is obtained from blurring a baseline T1 Gaussian FS [4], then all constrained embedded T1 FSs should have Gaussian MFs. Or, if we know an IT2 FS is constructed from the Interval Approach [7] or the Enhanced Interval Approach [2], then all constrained embedded T1 FSs should have triangular MFs. However, it is difficult to find "meaningful" constrained embedded T1 FSs for an arbitrary IT2 FS without priori information. Furthermore, sometimes it is impossible to cover the entire FOU of an IT2 FS using only "meaningful" embedded T1 FSs. An example is shown in Fig. 2. From the shape of the FOU we expect that "meaningful" embedded T1 FSs would have triangular or trapezoidal MFs. However, for any point within the more darkly shaded area in Fig. 2, it is impossible to find a normal triangle or trapezoid that passes through it, i.e., the complete FOU cannot be covered by only normal triangular or trapezoidal MFs.

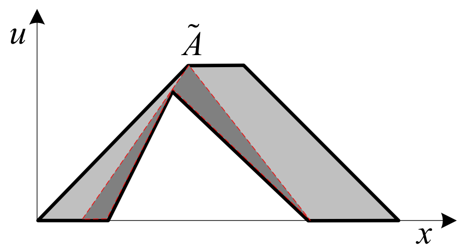


Fig. 2. A trapezoidal FOU which cannot be completely covered by "meaningful" normal triangular or trapezoidal embedded T1 FSs.

In this paper we propose a new Constrained Representation Theorem for IT2 FSs. It is more constrained than the Mendel-John Representation Theorem in that only convex and normal embedded T1 FSs are considered. However, it is more general than Garibaldi and Aisbett et al.'s idea in that we do not require the embedded T1 FSs to have a specific shape like Gaussian, triangular or trapezoid. The only requirements are convexity and normality. In this way, we can overcome their limitation, i.e., in our Constrained Representation Theorem the FOU of any convex and normal IT2 FS can be fully covered by only its convex and normal embedded T1 FSs. We also gave the algorithm for computing the constrained centroid of IT2 FSs using our Constrained Representation Theorem.

The rest of this paper is organized as follows: Section II introduces background knowledge on IT2 FSs and the Mendel-John Representation Theorem. Section III introduces the new Constrained Representation Theorem for IT2 FSs. Section IV presents the algorithm for computing the constrained centroid of an IT2 FS and two examples. Finally, Section V draws conclusions.

II. Background Knowledge on IT2 FSs and the Mendel-John Representation Theorem

This section presents background knowledge on IT2 FSs and the Mendel-John Representation Theorem.

A. Interval Type-2 Fuzzy Sets (IT2 FSs)

IT2 FSs are to-date the most widely used kind of type-2 FSs. An IT2 FS $\tilde{A}$ is described as[1]

$$\tilde{A} = \int_{x\in X}\int_{u\in J_x} 1/(x,u) = \int_{x\in X}\left[\int_{u\in J_x} 1/u\right]\Big/ x, \quad (1)$$

[1]This background material is taken from [11]. See also [9], [13].

where x is the *primary variable*, J_x, an interval in $[0,1]$, is the *primary membership* of x, u is the *secondary variable*, and $\int_{u\in J_x} 1/u$ is the *secondary MF* at x. Uncertainty about $\tilde{A}$ is conveyed by the union of all of the primary memberships, called the *footprint of uncertainty* of $\tilde{A}$ [$FOU(\tilde{A})$], i.e.,

$$FOU(\tilde{A}) = \bigcup_{x\in X} J_x \quad (2)$$

An IT2 FS is shown in Fig. 3. The FOU is shown as the shaded region. It is bounded by an *upper MF* (UMF) $\overline{\mu}_{\tilde{A}}(x)$ and a *lower MF* (LMF) $\underline{\mu}_{\tilde{A}}(x)$, both of which are T1 FSs; consequently, the membership grade of each element of an IT2 FS is an interval $[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]$.

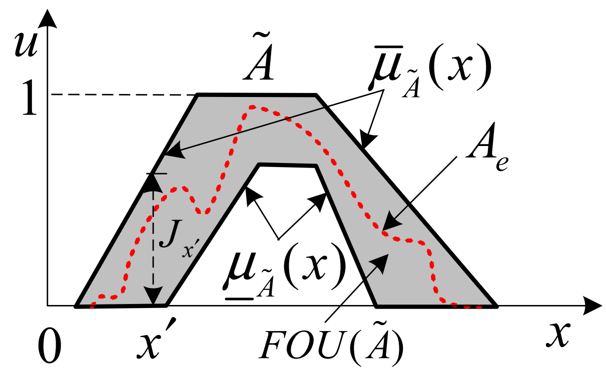


Fig. 3. An IT2 FS $\tilde{A}$. A_e is an embedded T1 FS defined in the Mendel-John Representation Theorem.

Note that an IT2 FS can also be represented as

$$\tilde{A} = 1/FOU(\tilde{A}) \quad (3)$$

with the understanding that this means putting a secondary grade of 1 at all points of $FOU(\tilde{A})$.

B. Mendel-John Representation Theorem

Mendel and John [12] have presented a Representation Theorem for general T2 FSs. Its special form for IT2 FS is introduced in this subsection. First, some necessary definitions are given.

Definition 1: [6] A T1 FS A is *convex* if and only if $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geqslant \min(\mu_A(x_1), \mu_A(x_2))$ for $\forall\, x_1, x_2 \in X$ and $\lambda \in [0,1]$. □

Definition 2: [13] A T1 FS A is *normal* if and only if $\sup_{x\in X}\mu_A(x) = 1$. □

Definition 3: An IT2 FS $\tilde{A}$ is *convex and normal* if and only if its UMF is convex and normal, and its LMF is convex. □

$\tilde{A}$ in Fig. 3 is convex and normal.

Definition 4: For discrete universes of discourse $X = \{x_1, x_2, \ldots, x_N\}$ and discrete J_x, an *embedded T1 FS* A_e has N elements, one each from $J_{x_1}, J_{x_2}, \ldots, J_{x_N}$, namely $u_1, u_2, \ldots, u_N$, i.e.

$$A_e = \sum_{i=1}^{N} u_i/x_i \qquad u_i \in J_{x_i} \subseteq [0,1]. \;\square \quad (4)$$

Note that if each J_{x_i} is discretized into M_i levels, there will be a total of n_A A_e, where

$$n_A = \prod_{i=1}^{N} M_i. \tag{5}$$

An example of A_e is shown in Fig. 3. Observe that it is not necessarily convex and normal.

Mendel-John Representation Theorem for IT2 FSs: Assume that primary variable x of an IT2 FS $\tilde{A}$ is sampled at N values, $x_1, x_2, \ldots, x_N$, and at each of these values its primary memberships u_i is sampled at M_i values, $u_{i1}, u_{i2}, \ldots, u_{iM_i}$. Let A_e^j denote the j^{th} embedded T1 FS for $\tilde{A}$. Then $\tilde{A}$ is represented by (3), in which[2]

$$\begin{aligned} FOU(\tilde{A}) &= \bigcup_{j=1}^{n_A} A_e^j \\ &= \bigcup_{x\in X} \left\{ \underline{\mu}_{\tilde{A}}(x), \ldots, \overline{\mu}_{\tilde{A}}(x) \right\} \\ &\equiv \bigcup_{x\in X} \left[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x) \right]. \quad \square \end{aligned} \tag{6}$$

In other words, this Representation Theorem states that the FOU of an IT2 FS is the union of all its embedded T1 FSs, including those that are non-convex and/or sub-normal. This implies that all these embedded T1 FSs should be considered in deriving theoretical results for IT2 FSs [16], e.g., uncertainty measures [21], [23], similarity measures [17], [18], [26], subsethood measures [17], [26], and the linguistic weighted average [13], [20], [22]. However, almost all applications of fuzzy logic systems use only convex and normal FSs; so, including non-convex and sub-normal embedded T1 FSs seems controversial. A Constrained Representation Theorem is proposed in the next section, where only convex and normal embedded T1 FSs are considered.

III. Constrained Representation Theorem for Convex and Normal IT2 FSs

In this section we propose a Constrained Representation Theorem for convex and normal IT2 FSs based on only convex and normal embedded T1 FSs (constrained embedded T1 FSs). We consider only convex and normal IT2 FSs, which are used in almost all applications of IT2 fuzzy logic systems.

Constrained Representation Theorem for Convex and Normal IT2 FSs: The FOU of a convex and normal IT2 FS is the union of all its convex and normal embedded T1 FSs. $\square$

For this Constrained Representation Theorem to be correct, we need to verify that the union of all convex and normal embedded T1 FSs can cover the entire FOU of a convex and normal IT2 FS, as indicated by the following:

[2] Although there are a finite number of embedded T1 FSs, it is customary to represent $FOU(\tilde{A})$ as an interval set $[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]$ at each x. Doing this is equivalent to discretizing with infinitesimally many small values and letting the discretizations approach zero.

Lemma 1: The FOU of a convex and normal IT2 FS can be completely covered by only its convex and normal embedded T1 FSs. $\square$

Proof: Consider an arbitrary point (p, h) within the FOU of a convex and normal IT2 FS, whose UMF has apex interval $[a, b]$, as shown in Fig. 4. We need to show that there is at least one convex and normal embedded T1 FS passing through (p, h). There can be only two cases:

1) $p < b$, as shown in Fig. 4(a): We can construct a convex and normal embedded T1 FS A_e, which starts from the LMF and then switches to the UMF at $x = p$.
2) $p \geqslant b$, as shown in Fig. 4(b): We can construct a convex and normal embedded T1 FS A_e, which starts from the UMF and then switches to the LMF at $x = p$.

In summary, for an arbitrary point within the FOU of a convex and normal IT2 FS, we can find at least one convex and normal embedded T1 FS which passes through it. So, the FOU of a convex and normal IT2 FS can be completely covered by only its convex and normal embedded T1 FSs. $\square$

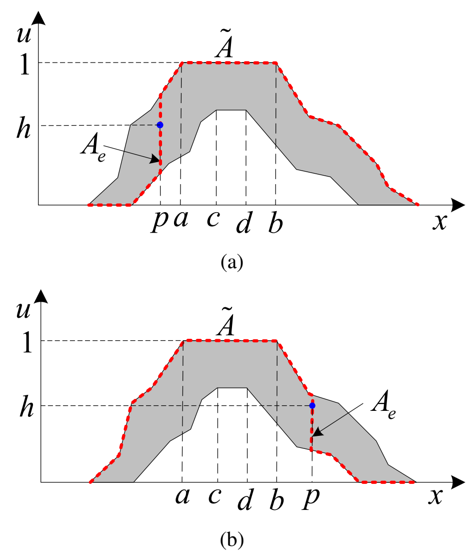


Fig. 4. Illustration of convex and normal embedded T1 FSs which pass through (p, h). (a) $p < b$; (b) $p \geqslant b$.

IV. Constrained Centroid of IT2 FSs Using the Constrained Representation Theorem

Centroid [5] may be the most important property of an IT2 FS. In this section we give the algorithm for computing the constrained centroid of an IT2 FS based on the Constrained Representation Theorem.

A. (Unconstrained) Centroid Based on the Mendel-John Representation Theorem

The centroid $c(A)$ of a T1 FS A is defined as

$$c(A) = \frac{\sum_{i=1}^{N} x_i \mu_A(x_i)}{\sum_{i=1}^{N} \mu_A(x_i)}. \tag{7}$$

Definition 5: [21] The *(unconstrained) centroid* $C(\tilde{A})$ of an IT2 FS $\tilde{A}$, computed based on the Mendel-John Representation Theorem, is the union of the centroids of all its (unconstrained) embedded T1 FSs A_e, i.e.,

$$C(\tilde{A}) \equiv \bigcup_{\forall A_e} c(A_e) = [c_l(\tilde{A}), c_r(\tilde{A})], \tag{8}$$

where $\bigcup$ is the union operation, and

$$c_l(\tilde{A}) = \min_{\forall A_e} c(A_e) \tag{9}$$

$$c_r(\tilde{A}) = \max_{\forall A_e} c(A_e). \quad \square \tag{10}$$

It has been shown [3], [5], [8], [9], [14] that $c_l(\tilde{A})$ and $c_r(\tilde{A})$ can be expressed as

$$c_l(\tilde{A}) = \min_{k\in[1,N-1]} \frac{\sum_{i=1}^{k} x_i \overline{\mu}_{\tilde{A}}(x_i) + \sum_{i=k+1}^{N} x_i \underline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{k} \overline{\mu}_{\tilde{A}}(x_i) + \sum_{i=k+1}^{N} \underline{\mu}_{\tilde{A}}(x_i)} \tag{11}$$

$$\equiv \frac{\sum_{i=1}^{L} x_i \overline{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^{N} x_i \underline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{L} \overline{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^{N} \underline{\mu}_{\tilde{A}}(x_i)} \tag{12}$$

$$c_r(\tilde{A}) = \max_{k\in[1,N-1]} \frac{\sum_{i=1}^{k} x_i \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=k+1}^{N} x_i \overline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{k} \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=k+1}^{N} \overline{\mu}_{\tilde{A}}(x_i)} \tag{13}$$

$$\equiv \frac{\sum_{i=1}^{R} x_i \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^{N} x_i \overline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{R} \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^{N} \overline{\mu}_{\tilde{A}}(x_i)}. \tag{14}$$

Switch points L and R, as well as $c_l(\tilde{A})$ and $c_r(\tilde{A})$, are traditionally computed by the KM or EKM algorithms [5], [9], [24]. Recently a much more efficient algorithm and its Matlab implementation were given in [19].

The main idea of the KM algorithms is to find the switch points for $c_l(\tilde{A})$ and $c_r(\tilde{A})$. Take $c_l(\tilde{A})$ as an example. $c_l(\tilde{A})$ is the minimum of $C(\tilde{A})$. So, we should choose a large weight [i.e., $\overline{\mu}_{\tilde{A}}(x_i)$] for small x_i and a small weight [i.e., $\underline{\mu}_{\tilde{A}}(x_i)$] for large x_i. The KM algorithm for $c_l(\tilde{A})$ finds the switch point L. For $i \leqslant L$, $\overline{\mu}_{\tilde{A}}(x_i)$ is used to calculate $c_l(\tilde{A})$; for $i > L$, $\underline{\mu}_{\tilde{A}}(x_i)$ is used. This ensures $c_l(\tilde{A})$ is the minimum.

B. Constrained Centroid Based on the Constrained Representation Theorem

In this subsection we introduce the algorithm for computing the constrained centroid of an IT2 FS based on the Constrained Representation Theorem.

Definition 6: The *constrained centroid* $C^c(\tilde{A})$ of an IT2 FS $\tilde{A}$ is the union of the centroids of all its *convex and normal* embedded T1 FSs A_e^{cn}, i.e.,

$$C^c(\tilde{A}) \equiv \bigcup_{\forall A_e^{cn}} c(A_e^{cn}) = [c_l^c(\tilde{A}), c_r^c(\tilde{A})], \tag{15}$$

where $\bigcup$ is the union operation, and

$$c_l^c(\tilde{A}) = \min_{\forall A_e^{cn}} c(A_e^{cn}) \tag{16}$$

$$c_r^c(\tilde{A}) = \max_{\forall A_e^{cn}} c(A_e^{cn}). \quad \square \tag{17}$$

Similar to the unconstrained case, for $c_l^c(\tilde{A})$ we still need a large weight for small x_i and a small weight for large x_i, i.e., the corresponding embedded T1 FS must still switch from the UMF to the LMF at some point. However, since A_e^{cn} must be convex and normal, we have two constraints:

1) Because A_e^{cn} must be normal, at least one point on it must have membership grade 1. So, the switch point L^c must satisfy $x_{L^c} \geqslant a$, where a is the left-most apex of the UMF of $\tilde{A}$, as shown in Fig. 5.
2) A_e^{cn} must also be convex. If $c \leqslant a$, as shown in Fig. 5(a), the convexity requirement is automatically satisfied when the normality constraints is satisfied, because all A_e^{cn} in the KM algorithm starts from the left-most point on the UMF, stays on the UMF until it reaches or passes the point $(a, 1)$, and then switches to and stays on the LMF. However, if $c > a$ and the switch point is between a and c, as shown in Fig. 5(b), then the MF of A_e^{cn} between a and c must be raised to h, the height of the LMF, to ensure that it is convex.

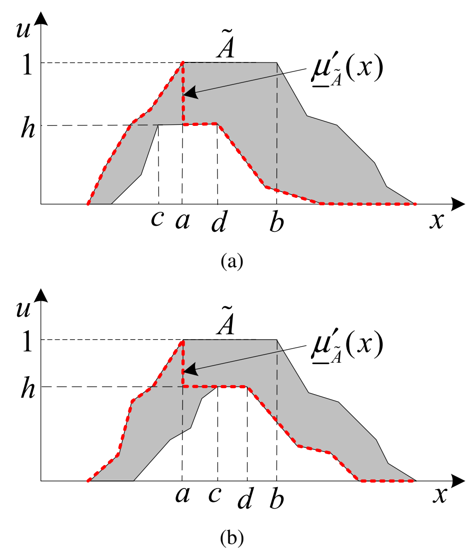


Fig. 5. $\underline{\mu}'_{\tilde{A}}(x)$, the LMF that should be used in the KM algorithm for computing $c_l^c(\tilde{A})$. (a) $c \leqslant a$; (b) $c > a$.

Though these two constraints seem complex, they can be simultaneously satisfied by smartly redefining the LMF of $\tilde{A}$ and then using it in the KM algorithm or its more efficient implementations. From the above analysis we know that when $c \leqslant a$, convexity is automatically satisfied when normality is satisfied, so we only need to worry about normality. If we re-define the LMF as

$$\underline{\mu}'_{\tilde{A}}(x_i) = \begin{cases} \overline{\mu}_{\tilde{A}}(x_i), & x_i \leqslant a \\ \underline{\mu}_{\tilde{A}}(x_i), & x_i > a \end{cases} \tag{18}$$

as shown in Fig. 5(a), then point $(a, 1)$ is guaranteed to be included in all embedded T1 FSs, and hence all embedded T1 FSs are convex and normal.

When $c > a$, we can re-define the LMF as

$$\underline{\mu}'_{\tilde{A}}(x_i) = \begin{cases} \overline{\mu}_{\tilde{A}}(x_i), & x_i \leqslant a \\ h, & a < x_i < c \\ \underline{\mu}_{\tilde{A}}(x_i), & x_i \geqslant c \end{cases} \tag{19}$$

as shown in Fig. 5(b). The motivation for defining $\underline{\mu}'_{\tilde{A}}(x_i) = \overline{\mu}_{\tilde{A}}(x_i)$ for $x_i \leqslant a$ in (19) is to ensure that the point $(a, 1)$ is included in every embedded T1 FS, i.e., every embedded T1 FS is normal. The motivation for defining $\underline{\mu}'_{\tilde{A}}(x_i) = h$ for $a < x_i < c$ is to ensure that every embedded T1 FS is also convex.

In summary, the following algorithm can be used to compute $c_l^c(\tilde{A})$:

1) If $c \leqslant a$, then re-define the LMF using (18); otherwise, re-define the LMF using (19).
2) Use the re-defined LMF and the original UMF in the KM algorithm or its more efficient implementations to compute $c_l^c(\tilde{A})$.

Similarly, to compute $c_r^c(\tilde{A})$, a small weight should be used for small x_i and a large weight for large x_i, i.e., the corresponding A_e^{cn} should still switch from the LMF to the UMF at some point. However, since this A_e^{cn} must be convex and normal, we again have two constraints:

1) Because A_e^{cn} must be normal, at least one point on it must have membership grade 1. So, the switch point R^c must satisfy $x_{R^c} \leqslant b$, where b is the right-most apex of the UMF of $\tilde{A}$, as shown in Fig. 6.
2) A_e^{cn} must also be convex. When $d \geqslant b$, as shown in Fig. 6(a), the convexity requirement is automatically satisfied when normality is satisfied, because all A_e^{cn} in the KM algorithm starts from the LMF and then switches to the UMF at or before the point $(b, 1)$. When $d < b$ and the switch happens between d and b, as shown in Fig. 6(b), the LMF between d and b must be raised to h, the height of the LMF, to ensure that it is convex.

Again, the two requirements can be simultaneously satisfied by smartly redefining the LMF of $\tilde{A}$ and then using it in the KM algorithm or its more efficient implementations. From the above analysis we know that when $d \geqslant b$, convexity is automatically satisfied when normality is satisfied, so we only need to worry about normality. If we re-define the LMF as

$$\underline{\mu}''_{\tilde{A}}(x_i) = \begin{cases} \underline{\mu}_{\tilde{A}}(x_i), & x_i < b \\ \overline{\mu}_{\tilde{A}}(x_i), & x_i \geqslant b \end{cases} \tag{20}$$

as shown in Fig. 6(a), then the point $(b, 1)$ is guaranteed to be included in all embedded T1 FSs, and hence all embedded T1 FSs are convex and normal.

When $d < b$, we can re-define the LMF as

$$\underline{\mu}''_{\tilde{A}}(x_i) = \begin{cases} \underline{\mu}_{\tilde{A}}(x_i), & x_i \leqslant d \\ h, & d < x_i < b \\ \overline{\mu}_{\tilde{A}}(x_i), & x_i \geqslant b \end{cases} \tag{21}$$

as shown in Fig. 6(b). The motivation for defining $\underline{\mu}''_{\tilde{A}}(x_i) = \overline{\mu}_{\tilde{A}}(x_i)$ for $x_i \geqslant b$ is to ensure that the point $(b, 1)$ is included in every embedded T1 FS, i.e., every embedded T1 FS is normal. The motivation for defining $\underline{\mu}''_{\tilde{A}}(x_i) = h$ for $d < x_i < b$ is to ensure that every embedded T1 FS is also convex.

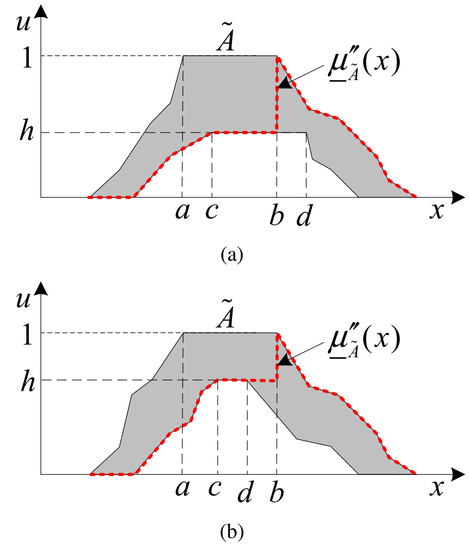


Fig. 6. $\underline{\mu}''_{\tilde{A}}(x)$, the LMF that should be used in the KM algorithm for computing $c_r^c(\tilde{A})$. (a) $d \geqslant b$; (b) $d < b$.

In summary, the following algorithm can be used to compute $c_r^c(\tilde{A})$:

1) If $d \geqslant b$, then re-define the LMF using (20); otherwise, re-define the LMF using (21).
2) Use the re-defined LMF and the original UMF in the KM algorithm or its more efficient implementations to compute $c_r^c(\tilde{A})$.

C. Some Properties of the Constrained Centroid

Because the (unconstrained) centroid of an IT2 FS is computed from its all possible embedded T1 FSs, regardless of whether they are convex and/or normal, whereas the constrained centroid is computed from only convex and normal embedded T1 FSs, the following result is true without proof:

Theorem 1: The constrained centroid computed based on the Constrained Representation Theorem is included in the (unconstrained) centroid computed based on the Mendel-John Representation Theorem. □

Another important property about the unconstrained centroid computed from the Mendel-John Representation Theorem is that $c_l(\tilde{A}) \to x_L$ and $c_r(\tilde{A}) \to x_R$ when $N \to \infty$ [15]. However, this property no longer holds for the constrained centroid.

D. Examples

Two examples are provided in this subsection to demonstrate the difference between the unconstrained centroid and the constrained centroid.

Example 1: Consider the trapezoidal FOU shown in Fig. 7(a). The domain of x, $[0, 6]$, was discretized into 1000 equally-spaced points in the computation, i.e. $N = 1000$. We obtained $C(\tilde{A}) = [2.67, 4.57]$ and $C^c(\tilde{A}) = [2.79, 4.56]$.

Observe that $C^c(\tilde{A}) \subset C(\tilde{A})$, which is consistent with Theorem 1. Observe also from Fig. 7 that the embedded T1 FSs in the constrained and unconstrained cases are quite different.

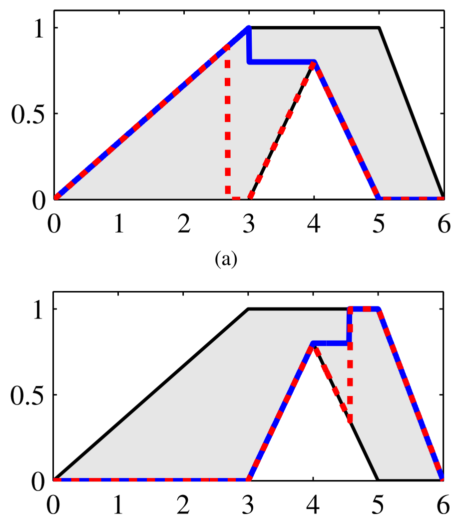


Fig. 7. The embedded T1 FSs determining (a) $c_l(\tilde{A})$ (red dashed curve) and $c_l^c(\tilde{A})$ (blue solid curve); and, (b) $c_r(\tilde{A})$ (red dashed curve) and $c_r^c(\tilde{A})$ (blue solid curve).

Example 2: Consider the Gaussian IT2 FS shown in Fig. 8(a), where the UMF is defined by $\exp(-x^2/2)$, and the LMF is defined by $0.5\exp(-x^2/2)$. Note that the IT2 FS is symmetrical about 0. So, according to the theoretical result in [10], $C(\tilde{A})$ must also be symmetrical about 0. In the computation the interval $[-3, 3]$ was discretized into 1000 equally-spaced points, i.e. $N = 1000$. The centroids are $C(\tilde{A}) = [-0.2737, 0.2737]$ and $C^c(\tilde{A}) = [-0.2637, 0.2637]$. Observe that:

1) $C(\tilde{A})$ is symmetrical about 0, as suggested by the theoretical result in [10].
2) $C^c(\tilde{A})$ is also symmetrical about 0. By following the proof in [10], we should also be able to prove that when an IT2 FS is symmetrical about m, its constrained centroid is also symmetrical about m.
3) $C^c(\tilde{A}) \subset C(\tilde{A})$, which is consistent with Theorem 1.
4) The embedded T1 FSs in the constrained and unconstrained cases are quite different, as shown in Fig. 8.

V. Conclusions

The Mendel-John Representation Theorem for IT2 FSs states that an IT2 FS is a combination of all its embedded T1 FSs, which can be non-convex and/or sub-normal. These non-convex and/or sub-normal embedded T1 FSs are included in developing many results for IT2 FSs, including the uncertainty measures and linguistic weighted average. However, almost all applications of fuzzy logic systems use only convex and normal FSs. In this paper, we have proposed a Constrained Representation Theorem for IT2 FSs using only convex and normal embedded T1 FSs. We also applied it to computing the centroid of IT2 FSs, the most important property and uncertainty measure of IT2 FSs. Our Constrained Representation Theorem can serve as the basis for developing many theoretical results, as the Mendel-John Representation Theorem has done.

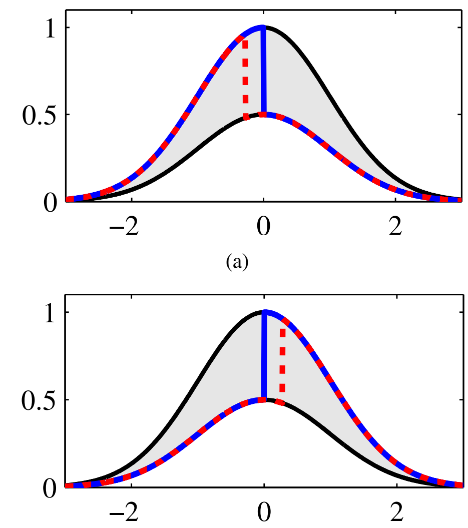


Fig. 8. The embedded T1 FSs determining (a) $c_l(\tilde{A})$ (red dashed curve) and $c_l^c(\tilde{A})$ (blue solid curve); and, (b) $c_r(\tilde{A})$ (red dashed curve) and $c_r^c(\tilde{A})$ (blue solid curve).

References

[1] J. Aisbett, J. T. Rickard, and D. Morgenthaler, "Multivariate modeling and type-2 fuzzy sets," *Fuzzy Sets and Systems*, vol. 163, no. 1, pp. 78–95, 2011.
[2] S. Coupland, J. M. Mendel, and D. Wu, "Enhanced interval approach for encoding words into interval type-2 fuzzy sets and convergence of the word fous," in *Proc. IEEE World Congress on Computational Intelligence*, July 2010.
[3] K. Duran, H. Bernal, and M. Melgarejo, "Improved iterative algorithm for computing the generalized centroid of an interval type-2 fuzzy set," in *Proc. Annual Meeting of the North American Fuzzy Information Processing Society*, New York, May 2008, pp. 1–5.
[4] J. M. Garibaldi, "Alternative forms of non-standard fuzzy sets: A discussion paper," in *Type-2 Fuzzy Logic: State of the Art and Future Directions*, London, June 2010.
[5] N. N. Karnik and J. M. Mendel, "Centroid of a type-2 fuzzy set," *Information Sciences*, vol. 132, pp. 195–220, 2001.
[6] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Upper Saddle River, NJ: Prentice-Hall, 1995.
[7] F. Liu and J. M. Mendel, "Encoding words into interval type-2 fuzzy sets using an interval approach," *IEEE Trans. on Fuzzy Systems*, vol. 16, no. 6, pp. 1503–1521, 2008.
[8] M. Melgarejo, "A fast recursive method to compute the generalized centroid of an interval type-2 fuzzy set," in *Proc. Annual Meeting of the North American Fuzzy Information Processing Society*, San Diego, CA, June 2007, pp. 190–194.
[9] J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. Upper Saddle River, NJ: Prentice-Hall, 2001.
[10] ——, "On a 50% savings in the computation of the centroid of a symmetrical interval type-2 fuzzy," *Information Sciences*, vol. 172, no. 3, pp. 417–430, 2005.
[11] J. M. Mendel, H. Hagras, and R. I. John, "Standard background material about interval type-2 fuzzy logic systems that can be used by all authors," http://ieee-cis.org/_files/standards.t2.win.pdf, accessed on Nov. 16, 2010.
[12] J. M. Mendel and R. I. John, "Type-2 fuzzy sets made simple," *IEEE Trans. on Fuzzy Systems*, vol. 10, no. 2, pp. 117–127, 2002.

[13] J. M. Mendel and D. Wu, *Perceptual Computing: Aiding People in Making Subjective Judgments*. Hoboken, NJ: Wiley-IEEE Press, 2010.

[14] J. M. Mendel and H. Wu, "New results about the centroid of an interval type-2 fuzzy set, including the centroid of a fuzzy granule," *Information Sciences*, vol. 177, pp. 360–377, 2007.

[15] ——, "Properties of the centroid of an interval type-2 fuzzy set, including the centroid of a fuzzy granule," in *Proc. IEEE Int'l Conf. on Fuzzy Systems*, Reno, NV, May 2005, pp. 341–346.

[16] J. Mendel, "On answering the question"where do i start in order to solve a new problem involving interval type-2 fuzzy sets?"," *Information Sciences*, vol. 179, no. 19, pp. 3418–3431, 2009.

[17] H. T. Nguyen and V. Kreinovich, "Computing degrees of subsethood and similarity for interval-valued fuzzy sets: Fast algorithms," in *Proc. 9th Int'l Conf. on Intelligent Technologies*, Samui, Thailand, October 2008, pp. 47–55.

[18] J. T. Rickard, J. Aisbett, and G. Gibbon, "Fuzzy subsethood for fuzzy sets of type-2 and generalized type-n," *IEEE Trans. on Fuzzy Systems*, vol. 17, no. 1, pp. 50–60, 2009.

[19] D. Wu, "Comparison and practical implementation of type-reduction algorithms for type-2 fuzzy sets and systems," submitted to World Conference on Soft Computing, San Francisco, CA, May 2011.

[20] D. Wu and J. M. Mendel, "Aggregation using the linguistic weighted average and interval type-2 fuzzy sets," *IEEE Trans. on Fuzzy Systems*, vol. 15, no. 6, pp. 1145–1161, 2007.

[21] ——, "Uncertainty measures for interval type-2 fuzzy sets," *Information Sciences*, vol. 177, no. 23, pp. 5378–5393, 2007.

[22] ——, "Corrections to 'Aggregation using the linguistic weighted average and interval type-2 fuzzy sets'," *IEEE Trans. on Fuzzy Systems*, vol. 16, no. 6, pp. 1664–1666, 2008.

[23] ——, "A comparative study of ranking methods, similarity measures and uncertainty measures for interval type-2 fuzzy sets," *Information Sciences*, vol. 179, no. 8, pp. 1169–1192, 2009.

[24] ——, "Enhanced Karnik-Mendel Algorithms," *IEEE Trans. on Fuzzy Systems*, vol. 17, no. 4, pp. 923–934, 2009.

[25] ——, "Computing with words for hierarchical decision making applied to evaluating a weapon system," *IEEE Trans. on Fuzzy Systems*, vol. 18, no. 3, pp. 441–460, 2010.

[26] ——, "Efficient algorithms for computing a class of subsethood and similarity measures for interval type-2 fuzzy sets," in *Proc. IEEE World Congress on Computational Intelligence*, Barcelona, Spain, July 2010.

[27] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-1," *Information Sciences*, vol. 8, pp. 199–249, 1975.