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# Uncertainty measures for interval type-2 fuzzy sets

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**Abstract**

Fuzziness (entropy) is a commonly used measure of uncertainty for type-1 fuzzy sets. For interval type-2 fuzzy sets (IT2 FSs), centroid, cardinality, fuzziness, variance and skewness are all measures of uncertainties. The centroid of an IT2 FS has been defined by Karnik and Mendel. In this paper, the other four concepts are defined. All definitions use a Representation Theorem for IT2 FSs. Formulas for computing the cardinality, fuzziness, variance and skewness of an IT2 FS are derived. These definitions should be useful in IT2 fuzzy logic systems design using the principles of uncertainty, and in measuring the similarity between two IT2 FSs.


*Keywords:* Uncertainty measure; Interval type-2 fuzzy set; Centroid; Cardinality; Fuzziness; Entropy; Variance; Skewness

## 1. Introduction

As pointed out by Zadeh [87], "*uncertainty is an attribute of information*". He proposed to use the generalized theory of uncertainty (GTU) to handle it. "*In GTU, uncertainty is linked to information through the concept of granular structure – a concept which plays a key role in human interaction with the real world [26,78,86]... Informally, a granule of a variable X is a clump of values of X which are drawn together by indistinguishability, equivalence, similarity, proximity or functionality. For example, an interval is a granule. So is a fuzzy interval...*"

To use fuzzy sets (FSs) as granules in GTU, there is a need to quantify the uncertainty associated with them. Klir [33] states that "*once uncertainty (and information) measures become well justified, they can very effectively be utilized for managing uncertainty and the associated information. For example, they can be utilized for extrapolating evidence, assessing the strength of relationship between given groups of variables, assessing the influence of given input variables on given output variables, measuring the loss of information when a system is simplified, and the like*".

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Three basic principles of uncertainty have been developed to guide the use of uncertainty measures in different situations [33,25]:

(1) The *principle of minimum uncertainty*, which states that solutions with the least loss of information should be selected, can be used in simplification and conflict resolution problems.
(2) The *principle of maximum uncertainty*, which states that a conclusion should maximize the relevant uncertainty within constraints given by the verified premises, is widely used within classical probability framework [14,15,56].
(3) The *principle of uncertainty invariance*, which states that the amount of uncertainty should be preserved in each transformation of uncertainty from one mathematical framework to another, is widely studied in the context of probability–possibility transformations [21,32,35,65].

However, as pointed out by Cross and Sudkamp [17], ''*the quantification of the degree of uncertainty in a FS depends upon the type of uncertainty one is trying to measure and on the particular measure selected for that type of uncertainty*''. Many uncertainty measures of type-1 (T1) FSs have been proposed. Among them, *fuzziness* (entropy) [17,34] is frequently used, and it will be studied in this paper.

In addition to fuzziness, *centroid, cardinality, variance* and *skewness* are also important characteristics of T1 FSs. For example, Dubois and Prade [19] point out that ''*cardinality is a natural tool for capturing the meaning of linguistic quantifiers* [80–85,77] *and to provide satisfactory answers to queries pertaining to quantification, of the form* '*How many X's are A*' , '*Are there more X's which are A than X's which are B,*' *etc*''. These queries [74] ''*occur in computing with words, communication with data bases and information/intelligent systems, modeling the meaning of imprecise quantifiers in natural language statements, decision-making in a fuzzy environment, analysis of grey-tone images, clustering, etc*''. These four characteristics can also be used to measure the distance or similarity between two T1 FSs. For example, Wenstøp [64] uses the centroid and the cardinality of T1 FSs to measure their distance. This enables one FS to be found from a group of T1 FSs $B_i$ $(i = 1, \ldots, N)$ that most resembles a target T1 FS $A$. Bonissone [6,7] uses a two-step approach to solve the same problem. In his first step, four measures – centroid, cardinality, fuzziness and skewness – are used to identify several FSs from the $N$ $B_i$ which are close to $A$.

Recently, there has been a growing interest in type-2 (T2) fuzzy set and system theory [79,46,47]. The membership grades of a T2 FS are T1 FSs in $[0, 1]$ instead of crisp numbers. Since the boundaries of T2 FSs are blurred, they are especially useful in circumstances where it is difficult to determine an exact membership grade [46]. To date, interval T2 (IT2) FSs are the most widely used T2 FSs, and have been used successfully for decision making [76,55,59,66], time-series forecasting [46,4], survey processing [46,3,42], document retrieval [8], speech recognition [88,45], noise cancellation [12,54], word modeling [50,72,42], clustering [57], control [71,70,38,24,13,43,20,41,58,11,44,1], wireless communication [40,60], webshopping [23], linguistic summarization of database [53,52], etc.

Though the above applications have demonstrated that IT2 FSs are better at modeling uncertainties than T1 FSs, uncertainty measures for IT2 FSs have not been extensively studied. Centroid, cardinality, fuzziness, variance and skewness are all uncertainty measures for IT2 FSs because each of them is an interval (see Section 3), and the length of the interval is an indicator of uncertainty, i.e. the larger (smaller) the interval, the more (less) the uncertainty. Once these uncertainty measures are defined for IT2 FSs, their applications in T1 FSs can be extended to IT2 FSs, e.g. the centroid and cardinality of IT2 FSs have been used in [69] to define a vector similarity measure for IT2 FSs.

The centroid of an IT2 FS has been well-defined and studied by Karnik and Mendel [28]. Because the centroid of an IT2 FS has no closed-form solution, they developed iterative algorithms, now called Karnik–Mendel (KM) Algorithms, to compute it. The cardinality of an IT2 FS was introduced in [69]. For completeness, the centroid and cardinality are again introduced in this paper. Additionally, the other three uncertainty measures of IT2 FSs – fuzziness, variance and skewness – are defined and shown how to be computed.

The rest of this paper is organized as follows: Section 2 provides background material. Section 3 gives definitions of centroid, cardinality, fuzziness, variance and skewness for IT2 FSs, and explains how to compute them. Section 4 draws conclusions. The KM Algorithms are given in the Appendix.

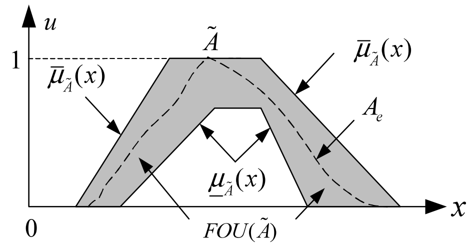


Fig. 1. An IT2 FS. $A_e$ is an embedded T1 FS.

## 2. Background

### 2.1. Interval type-2 fuzzy sets (IT2 FSs)

An IT2 FS, $\widetilde{A}$, is to-date the most widely used kind of T2 FS, and is the only kind of T2 FS that is considered in this paper. It is described as[1]

$$\widetilde{A} = \int_{x\in X}\int_{u\in J_x} 1/(x,u) = \int_{x\in X}\left[\int_{u\in J_x} 1/u\right]\Big/ x, \tag{1}$$

where $x$ is the *primary variable*, $J_x$, an interval in $[0,1]$, is the *primary membership* of $x$, $u$ is the *secondary variable*, and $\int_{u\in J_x} 1/u$ is the *secondary membership function* (MF) at $x$. Note that (1) means: $\widetilde{A} : X \to \{[a,b] : 0 \leqslant a \leqslant b \leqslant 1\}$. Uncertainty about $\widetilde{A}$ is conveyed by the union of all of the primary memberships, called the *footprint of uncertainty* of $\widetilde{A}$ [FOU($\widetilde{A}$)], i.e.

$$\mathrm{FOU}(\widetilde{A}) = \bigcup_{x\in X} J_x. \tag{2}$$

An IT2 FS is shown in Fig. 1. The FOU is shown as the shaded region. It is bounded by an *upper MF* (UMF) $\overline{\mu}_{\widetilde{A}}(x)$ and a *lower MF* (LMF) $\underline{\mu}_{\widetilde{A}}(x)$, both of which are T1 FSs; consequently, the membership grade of each element of an IT2 FS is an interval $[\underline{\mu}_{\widetilde{A}}(x), \overline{\mu}_{\widetilde{A}}(x)]$.

Note that an IT2 FS can also be represented as

$$\widetilde{A} = 1/\mathrm{FOU}(\widetilde{A}) \tag{3}$$

with the understanding that this means putting a secondary grade of 1 at all points of FOU($\widetilde{A}$).

For discrete universes of discourse $X = \{x_1, x_2, \ldots, x_N\}$ and discrete $J_x$, an *embedded T1 FS* $A_e$ has $N$ elements, one each from $J_{x_1}, J_{x_2}, \ldots, J_{x_N}$, namely $u_1, u_2, \ldots, u_N$, i.e.

$$A_e = \sum_{i=1}^{N} u_i/x_i \quad u_i \in J_{x_i} \subseteq [0,1]. \tag{4}$$

Examples of $A_e$ are $\overline{\mu}_{\widetilde{A}}(x)$ and $\underline{\mu}_{\widetilde{A}}(x)$; see, also Fig. 1. Note that if each $J_{x_i}$ is discretized into $M_i$ levels, there will be a total of $n_A A_e$, where

$$n_A = \prod_{i=1}^{N} M_i. \tag{5}$$

### 2.2. Representation theorem

Mendel and John [49] have presented a Representation Theorem for a general T2 FS, which when specialized to an IT2 FS can be expressed as:

[1] This background material is taken from [48]. See also [46].

**Representation Theorem for an IT2 FS:** Assume that primary variable $x$ of an IT2 FS $\widetilde{A}$ is sampled at $N$ values, $x_1, x_2, \ldots, x_N$, and at each of these values its primary memberships $u_i$ are sampled at $M_i$ values, $u_{i1}, u_{i2}, \ldots, u_{iM_i}$. Let $A_e^j$ denote the $j$th embedded T1 FS for $\widetilde{A}$. Then $\widetilde{A}$ is represented by (3), in which[2]

$$\mathrm{FOU}(\widetilde{A}) = \bigcup_{j=1}^{n_A} A_e^j = \bigcup_{x\in X}\{\underline{\mu}_{\widetilde{A}}(x), \ldots, \overline{\mu}_{\widetilde{A}}(x)\} \equiv \bigcup_{x\in X}[\underline{\mu}_{\widetilde{A}}(x), \overline{\mu}_{\widetilde{A}}(x)]. \tag{6}$$

This representation of an IT2 FS, in terms of simple T1 FSs, the embedded T1 FSs, is very useful for deriving theoretical results; however, it is not recommended for computational purposes, because it would require the enumeration of the $n_A$ embedded T1 FSs and $n_A$ (given in (5)) can be astronomical. The Representation Theorem will be used heavily in defining the centroid, cardinality, fuzziness, variance and skewness of IT2 FSs.

## 3. Uncertainty measures for IT2 FSs

In this section T1 FS definitions of cardinality, fuzziness, variance and skewness are extended to IT2 FSs.[3] Because defining the variance and skewness of an IT2 FS uses its centroid, the definition of the centroid of an IT2 FS is reviewed first. Additionally, because discrete versions of these definitions are more frequently used in practice, and one can easily deduce the corresponding continuous versions of these definitions from the discrete versions, only discrete cases are considered in this paper.

As stated in Section 1, centroid, cardinality, fuzziness, variance and skewness are uncertainty measures for IT2 FSs because each of them is an interval, and the length of the interval is an indicator of uncertainty.

### 3.1. Centroid of an IT2 FS

The centroid $c(A)$ of the T1 FS $A$ is defined as

$$c(A) = \frac{\sum_{i=1}^{N} x_i \mu_A(x_i)}{\sum_{i=1}^{N} \mu_A(x_i)}. \tag{7}$$

**Definition 1.** The centroid $C_{\widetilde{A}}$ of an IT2 FS $\widetilde{A}$ is the union of the centroids of all its embedded T1 FSs $A_e$, i.e.,

$$C_{\widetilde{A}} \equiv \bigcup_{\forall A_e} c(A_e) = [c_l(\widetilde{A}), c_r(\widetilde{A})], \tag{8}$$

where $\bigcup$ is the union operation, and

$$c_l(\tilde{A}) = \min_{\forall A_e} c(A_e), \tag{9}$$

$$c_r(\tilde{A}) = \max_{\forall A_e} c(A_e). \tag{10}$$

It has been shown [28,46,51] that $c_l(\tilde{A})$ and $c_r(\tilde{A})$ can be expressed as

$$c_l(\tilde{A}) = \frac{\sum_{i=1}^{L} x_i \overline{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^{N} x_i \underline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{L} \overline{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^{N} \underline{\mu}_{\tilde{A}}(x_i)}, \tag{11}$$

$$c_r(\tilde{A}) = \frac{\sum_{i=1}^{R} x_i \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^{N} x_i \overline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{R} \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^{N} \overline{\mu}_{\tilde{A}}(x_i)}. \tag{12}$$

Switch points $x_L$ and $x_R$, as well as $c_l(\widetilde{A})$ and $c_r(\widetilde{A})$, are computed by using the iterative KM Algorithms [46,28] given in the Appendix.

[2] Although there are a finite number of embedded T1 FSs, it is customary to represent $\mathrm{FOU}(\widetilde{A})$ as an interval set $[\underline{\mu}_{\widetilde{A}}(x), \overline{\mu}_{\widetilde{A}}(x)]$ at each $x$. Doing this is equivalent to discretizing with infinitesimally many small values and letting the discretizations approach zero.

[3] The centroid of an IT2 FS has been well-defined by Karnik and Mendel [28] and Mendel [46]. A continuous version definition of the cardinality of an IT2 FS was introduced in [69]. In this paper a discrete version definition of the cardinality is introduced.

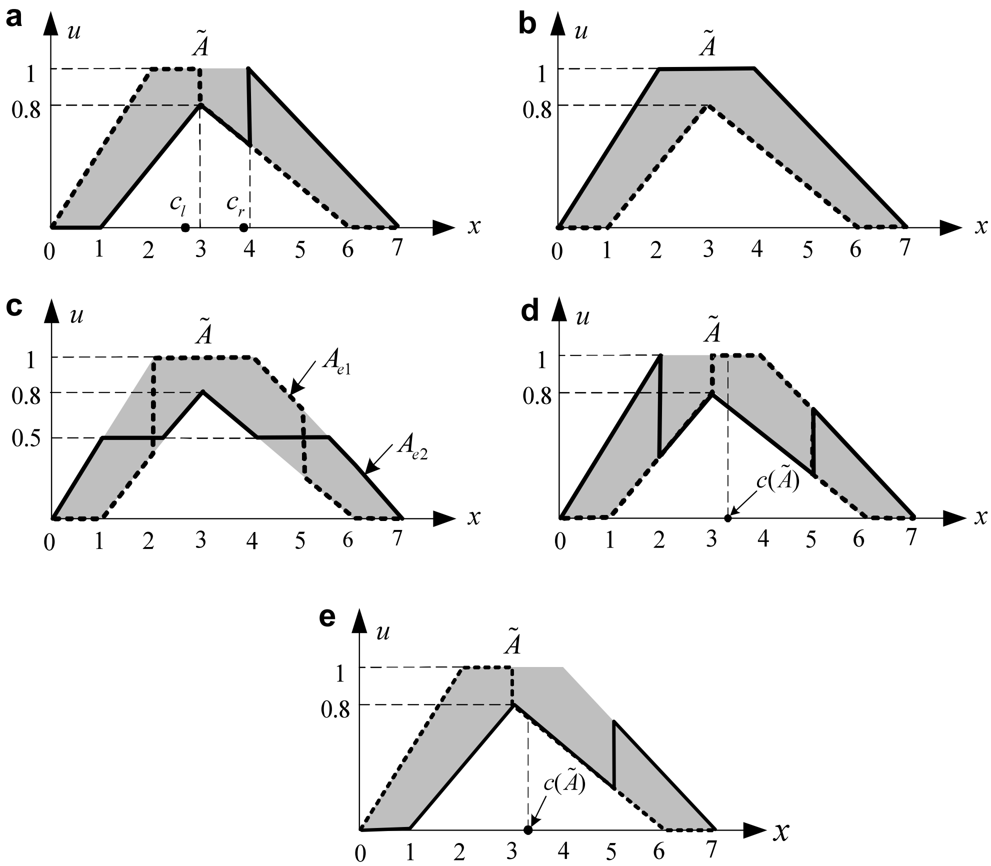


Fig. 2. The embedded T1 FSs determining (a) centroid, (b) cardinality, (c) fuzziness (entropy), (d) variance, and (e) skewness of an IT2 FS $\widetilde{A}$. In each figure, the dashed curve determines the left bound of the corresponding uncertainty measure, and the solid curve determines the right bound.

**Example 1.** [4]Consider the FOU shown in Fig. 2a. The domain of $x$, $[0,7]$, was discretized into 8 equally-spaced points in the computation, i.e. $N=8$. Note that $N=8$ is only for illustrative purpose; in practice $N$ is usually chosen to be much larger so that the results are more accurate. In this example, $C_{\tilde{A}} = [2.70, 3.92]$.

Observe from Fig. 2a that

(1) The embedded T1 FS determining $c_l(\widetilde{A})$ switches from the UMF of $\widetilde{A}$ to the LMF as $x$ increases, whereas the embedded T1 FS determining $c_r(\widetilde{A})$ switches from the LMF to the UMF as $x$ increases.
(2) The embedded T1 FS determining $c_l(\widetilde{A})$ switches from the UMF to the LMF at $x_L = 3$, whereas $c_l(\widetilde{A}) = 2.70$. Similarly, $x_R = 4$ whereas $c_r(\widetilde{A}) = 3.92$. $c_l(\widetilde{A}) \neq x_L$ and $c_r(\widetilde{A}) \neq x_R$ because discretization is used. For the continuous case, we always have $c_l(\widetilde{A}) = x_L$ and $c_r(\widetilde{A}) = x_R$ [51].
(3) Generally the two embedded T1 FSs determining $c_l(\widetilde{A})$ and $c_r(\widetilde{A})$ are not convex.

### 3.2. Cardinality of an IT2 FS

Definitions of the cardinality of T1 FSs have been proposed by several authors, e.g. [18,30,22,82,5,31,73], etc. Basically there are two kinds of proposals [19,74]: (1) those which assume that the cardinality of a T1 FS

[4] Simple examples are used in this paper so that the embedded T1 FSs associated with the bounds of each uncertainty measure can be shown clearly.

could be a precise number; and, (2) those which claim that it should be a fuzzy integer. De Luca and Termini's [18] definition of cardinality, also called the *power* of a T1 FS, is the sum of all membership grades, i.e.

$$p_{\mathrm{DT}}(A) = \sum_{i=1}^{N} \mu_A(x_i). \tag{13}$$

(13) is the most frequently used definition of cardinality; however, $p_{\mathrm{DT}}(A)$ increases as $N$ increases, and $\lim_{N\to\infty} p_{\mathrm{DT}}(A)$ does not exist. In this paper we define a *normalized cardinality* for a T1 FS by discretizing De Luca and Termini's cardinality definition in the continuous domain, $\int_X \mu_A(x)\mathrm{d}x$, i.e.

$$p(A) = \frac{|X|}{N} \sum_{i=1}^{N} \mu_A(x_i), \tag{14}$$

where $|X| = x_N - x_1$ is the length of the universe of discourse used in the computation. $X$ can be part of the complete universe of discourse because for some MFs (e.g., Gaussian, Bell) the complete universes of discourse are infinite. Usually $x_i$ $(i = 1, 2, \ldots, N)$ are chosen equally-spaced in the domain of $x$; in this case, $p(A)$ converges to its continuous version, $\int_X \mu_A(x)\mathrm{d}x$, as $N$ increases.

The cardinality of T2 FSs has not been studied by many researchers. Jang and Ralescu [27] defined a fuzzy-valued cardinality of a FS-valued function, which can be viewed as a general T2 FS. Szmidt and Kacprzyk [62] derived an interval cardinality for intuitionistic fuzzy sets (IFS). Though IFSs are different from IT2 FSs, Atanassov and Gargov [2] showed that every IFS can be mapped to an interval valued FS, which is an IT2 FS under a different name. Using Atanassov and Gargov's mapping, Szmidt and Kacprzyk's interval cardinality for an IT2 FS $\widetilde{A}$ is

$$P_{\mathrm{SK}}(\widetilde{A}) = \left[\min_{\forall A_e} p_{\mathrm{DT}}(A_e), \max_{\forall A_e} p_{\mathrm{DT}}(A_e)\right] \equiv [p_{\mathrm{DT}}(\underline{\mu}_{\tilde{A}}), p_{\mathrm{DT}}(\overline{\mu}_{\tilde{A}})]. \tag{15}$$

Note that (15) is defined based on (13). In the following an interval cardinality for an IT2 FS is defined based on (14).

**Definition 2.** The cardinality of an IT2 FS $\widetilde{A}$ is the union of all cardinalities of its embedded T1 FSs $A_e$, i.e.,

$$P_{\tilde{A}} \equiv \bigcup_{\forall A_e} p(A_e) = [p_{\mathrm{l}}(\widetilde{A}), p_{\mathrm{r}}(\widetilde{A})], \tag{16}$$

where

$$p_{\mathrm{l}}(\widetilde{A}) = \min_{\forall A_e} p(A_e), \tag{17}$$

$$p_{\mathrm{r}}(\widetilde{A}) = \max_{\forall A_e} p(A_e). \tag{18}$$

Note that this definition is quite similar to Szmidt and Kacprzyk's (see (15)). The only difference is that a different T1 cardinality measure is used in (16).

**Theorem 1.** *$p_{\mathrm{l}}(\widetilde{A})$ and $p_{\mathrm{r}}(\widetilde{A})$ in (17) and (18) can be computed as*

$$p_{\mathrm{l}}(\widetilde{A}) = p(\underline{\mu}_{\tilde{A}}(x)), \tag{19}$$

$$p_{\mathrm{r}}(\widetilde{A}) = p(\overline{\mu}_{\tilde{A}}(x)). \tag{20}$$

**Proof.** The proof is quite simple, and is

$$\begin{aligned} p_{\mathrm{l}}(\widetilde{A}) &= \min_{\forall A_e} p(A_e) = \min_{\forall A_e}\left[\frac{|X|}{N}\sum_{i=1}^{N}\mu_{A_e}(x_i)\right] = \frac{|X|}{N}\sum_{i=1}^{N}\left[\min_{\forall A_e}\mu_{A_e}(x_i)\right] \\ &= \frac{|X|}{N}\sum_{i=1}^{N}\underline{\mu}_{\tilde{A}}(x_i) = p(\underline{\mu}_{\tilde{A}}(x)) \end{aligned} \tag{21}$$

$$p_r(\widetilde{A}) = \max_{\forall A_e} p(A_e) = \max_{\forall A_e} \left[ \frac{|X|}{N} \sum_{i=1}^{N} \mu_{A_e}(x_i) \right] = \frac{|X|}{N} \sum_{i=1}^{N} \left[ \max_{\forall A_e} \mu_{A_e}(x_i) \right]$$
$$= \frac{|X|}{N} \sum_{i=1}^{N} \overline{\mu}_{\tilde{A}}(x_i) = p(\overline{\mu}_{\tilde{A}}(x)). \qquad \square \tag{22}$$

Another useful concept is the *average cardinality* of $\widetilde{A}$, which is defined as the average of its minimum and maximum cardinalities, i.e.,

$$\mathrm{AC}(\widetilde{A}) = \frac{p(\underline{\mu}_{\tilde{A}}(x)) + p(\overline{\mu}_{\tilde{A}}(x))}{2}. \tag{23}$$

$\mathrm{AC}(\widetilde{A})$ has been used in [69] to define a vector similarity measure for IT2 FSs. Note that Vlachos and Sergiadis [63] have defined an *average possible cardinality* in the similar manner, except that $p_{\mathrm{DT}}(\underline{\mu}_{\tilde{A}}(x))$ and $p_{\mathrm{DT}}(\overline{\mu}_{\tilde{A}}(x))$ were used in the numerator of (23).

**Example 2.** For the IT2 FS $\widetilde{A}$ shown in Fig. 2b, which is the same as the one shown in Fig. 2a, $P_{\tilde{A}} = [1.75, 3.92]$ and $\mathrm{AC}(\widetilde{A}) = 2.84$. Observe from Fig. 2b that $P_{\tilde{A}}$ is completely determined by the LMF and UMF of $\tilde{A}$.

### 3.3. Fuzziness (entropy) of an IT2 FS

The fuzziness (entropy) of a T1 FS is used to quantify the amount of vagueness in it. A T1 FS $C$ is most fuzzy when all its memberships equal 0.5. A T1 FS $A$ is more fuzzy than a T1 FS $B$ if $A$ is nearer to such a $C$ than $B$ is.

**Example 3.** In Fig. 3, $A$ is more fuzzy than $B$ because the memberships of $A$ are closer to $\mu = 0.5$.

Many different fuzziness measures have been proposed [34] for T1 FSs. Three of them are summarized in Table 1. It is straightforward to show that all of them are special cases of a *general fuzziness measure* [37].

**Definition 3.** A general fuzziness measure of a T1 FS $A$, $f(A)$, is defined as

$$f(A) = h\left(\sum_{i=1}^{N} g(\mu_A(x_i))\right), \tag{24}$$

where $h$ is a monotonically increasing function from $R^+$ to $R^+$, and, $g : [0,1] \to R^+$ is a function associated with each $x_i$. Additionally, (1) $g(0) = g(1) = 0$; (2) $g(0.5)$ is a unique maximum of $g$; and, (3) $g$ must be monotonically increasing on $[0, 0.5]$ and monotonically decreasing on $[0.5, 1]$.

Theoretically, $f(A)$ may be any T1 fuzziness definition satisfying the requirements in Definition 3; however, a normalized version such as Yager's definition is preferred by us because it converges as $N$ increases.

Several researchers have proposed definitions of the fuzziness for IT2 FSs, as summarized in Table 2. Note that Szmidt and Kacprzyk's [62] definition and Cornelis and Kerre's [16] definition are proposed for IFSs.

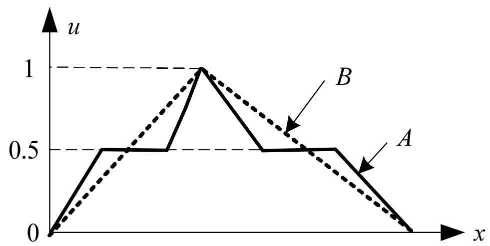


Fig. 3. $A$ (solid lines) is more fuzzy than $B$ (dashed lines).

Table 1
Three fuzziness (entropy) measures for T1 FSs

| Authors | Formulas |
|---|---|
| De Luca and Termini [18] | $f_{DT}(A) = -\sum_{i=1}^{N}[\mu_A(x_i)\log_2(\mu_A(x_i)) + (1-\mu_A(x_i))\log_2(1-\mu_A(x_i))]$ |
| Kaufmann [29] | $f_K(A) = \left[\sum_{i=1}^{N}\lvert\mu_A(x_i) - \mu_{A_{\text{near}}}(x_i)\rvert^r\right]^{\frac{1}{r}}$, where $\mu_{A_{\text{near}}}(x) = \begin{cases} 0, & \text{if } \mu_A \leqslant 0.5,\\ 1, & \text{otherwise.}\end{cases}$ |
| Yager [75] | $f_Y(A) = 1 - \frac{\left[\sum_{i=1}^{N}\lvert 2\mu_A(x_i)-1)\rvert^r\right]^{\frac{1}{r}}}{N^{\frac{1}{r}}}$, where $r$ is a positive constant. |

Table 2
Five existing fuzziness (entropy) measures for IT2 FSs

| Authors | Formulas |
|---|---|
| Burillo and Bustince [9] | $F_{BB}(\tilde{A}) = \sum_{i=1}^{N}[\overline{\mu}_{\tilde{A}}(x_i) - \underline{\mu}_{\tilde{A}}(x_i)]$ |
| Szmidt and Kacprzyk [62] | $F_{SK}(\tilde{A}) = \frac{1}{N}\sum_{i=1}^{N}\frac{1-\max[1-\overline{\mu}_{\tilde{A}}(x_i),\underline{\mu}_{\tilde{A}}(x_i)]}{1-\min[1-\overline{\mu}_{\tilde{A}}(x_i),\underline{\mu}_{\tilde{A}}(x_i)]}$ |
| Zeng and Li [89] | $F_{ZL}(\tilde{A}) = 1 - \frac{1}{N}\sum_{i=1}^{N}\lvert\overline{\mu}_{\tilde{A}}(x_i) + \underline{\mu}_{\tilde{A}}(x_i) - 1\rvert$ |
| Vlachos and Sergiadis [63] | $F_{VS}(\tilde{A}) = \frac{p(\tilde{A}\cap\tilde{A}^c)}{p(\tilde{A}\cup\tilde{A}^c)}$, where $\widetilde{A}^c$ is the complementary set of $\tilde{A}$. |
| Cornelis and Kerre [16] | $F_{CK}(\tilde{A}) = \left[\frac{2}{N}\sum_{i=1}^{N}\min(\underline{\mu}_{\tilde{A}}(x_i), 1-\overline{\mu}_{\tilde{A}}(x)), \frac{2}{N}\sum_{i=1}^{N}\min(0.5, 1-\underline{\mu}_{\tilde{A}}(x_i), \overline{\mu}_{\tilde{A}}(x))\right]$ |

They are converted to fuzziness measures for IT2 FSs by Atanassov and Gargov's [2] mapping. Note also that the first four methods give crisp measures, whereas the last one gives an interval measure.

In the following an interval fuzziness definition based on the Mendel–John Representation Theorem is proposed.

**Definition 4.** The fuzziness $F_{\tilde{A}}$ of an IT2 FS $\widetilde{A}$ is the union of the fuzziness of all its embedded T1 FSs $A_e$, i.e.,

$$F_{\tilde{A}} \equiv \bigcup_{\forall A_e} f(A_e) = [f_l(\widetilde{A}), f_r(\widetilde{A})], \tag{25}$$

where $f_l(\widetilde{A})$ and $f_r(\widetilde{A})$ are the minimum and maximum of the fuzziness of all $A_e$, respectively, i.e.

$$f_l(\widetilde{A}) = \min_{\forall A_e} f(A_e), \tag{26}$$

$$f_r(\widetilde{A}) = \max_{\forall A_e} f(A_e), \tag{27}$$

and $f(A_e)$ satisfies Definition 3.

**Theorem 2.** *Let $A_{e1}$ be defined as*

$$\mu_{A_{e1}}(x) = \begin{cases} \overline{\mu}_{\tilde{A}}(x), & \overline{\mu}_{\tilde{A}}(x) \text{ is further away from } 0.5 \text{ than } \underline{\mu}_{\tilde{A}}(x),\\ \underline{\mu}_{\tilde{A}}(x), & \text{otherwise,}\end{cases} \tag{28}$$

*and $A_{e2}$ be defined as*

$$\mu_{A_{e2}}(x) = \begin{cases} \overline{\mu}_{\tilde{A}}(x), & \text{both } \overline{\mu}_{\tilde{A}}(x) \text{ and } \underline{\mu}_{\tilde{A}}(x) \text{ are below } 0.5,\\ \underline{\mu}_{\tilde{A}}(x), & \text{both } \overline{\mu}_{\tilde{A}}(x) \text{ and } \underline{\mu}_{\tilde{A}}(x) \text{ are above } 0.5,\\ 0.5, & \text{otherwise.}\end{cases} \tag{29}$$

*Then (26) and (27) can be computed as*

$$f_l(\widetilde{A}) = f(A_{e1}), \tag{30}$$

$$f_r(\widetilde{A}) = f(A_{e2}), \tag{31}$$

*where $f(A)$ is defined in (24).*

**Proof.** According to Definition 3, the further away $\mu_{A_e}(x)$ is from 0.5, the smaller the fuzziness is; and the closer $\mu_{A_e}(x)$ is to 0.5, the larger the fuzziness is. Consequently, $f_l(\widetilde{A})$ is achieved when every $\mu_{A_{e1}}(x)$ is as far away as possible from 0.5, and $f_r(\widetilde{A})$ is achieved when every $\mu_{A_{e2}}(x)$ is as close as possible to 0.5.

Because $\mu_{A_e}(x) \in [\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]$, $\mu_{A_e}(x)$ furthest away from 0.5 can only be achieved at the two boundaries, $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$; hence, we only need to compare which of them is further away from 0.5. If $\overline{\mu}_{\tilde{A}}(x)$ is further away from 0.5 than $\underline{\mu}_{\tilde{A}}(x)$, we should set $\mu_{A_{e1}}(x) = \overline{\mu}_{\tilde{A}}(x)$; otherwise, we set $\mu_{A_{e1}}(x) = \underline{\mu}_{\tilde{A}}(x)$. This proves (30).

The proof of (31) is also straightforward. When the entire interval $[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]$ is below 0.5, $\overline{\mu}_{\tilde{A}}(x)$ is closest to 0.5, so we should set $\mu_{A_{e2}}(x) = \overline{\mu}_{\tilde{A}}(x)$; when the entire interval $[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]$ is above 0.5, $\underline{\mu}_{\tilde{A}}(x)$ is closest to 0.5, so we should set $\mu_{A_{e2}}(x) = \underline{\mu}_{\tilde{A}}(x)$; finally, when $0.5 \in [\underline{\mu}_{\tilde{A}}(x), \underline{\mu}_{\tilde{A}}(x)]$, we set $\mu_{A_{e2}}(x) = 0.5$. □

Observe that Cornelis and Kerre's fuzziness measure [16] (see Table 2) is a special case of (25) when $f(A) = \frac{2}{N}\sum_{i=1}^{N} A(x_i) \cap \bar{A}(x_i)$. The two embedded T1 FSs determining the left and right bounds of $F_{CK}(\widetilde{A})$ are the same as $A_{e1}$ and $A_{e2}$ in Theorem 2.

**Example 4.** Consider the IT2 FS $\widetilde{A}$ in Fig. 2c, which is the same as the IT2 FS shown in Fig. 2a. According to (28) and (29), $A_{e1}$ and $A_{e2}$ are as shown in Fig. 2c. When Yager's definition is used and $r = 1$, $F_{\tilde{A}} = [0.07, 0.63]$.

Observe from Fig. 2c that both $A_{e1}$ and $A_{e2}$ may switch between the LMF and UMF of $\widetilde{A}$, and $A_{e2}$ may have portions that belong to neither the LMF nor the UMF.

### 3.4. Variance of an IT2 FS

The variance of a T1 FS $A$ measures its compactness, i.e. a smaller (larger) variance means $A$ is more (less) compact.

**Example 5.** In Fig. 4, $A$ has smaller variance than $B$ because it is more compact.

One definition of the (possibilistic) variance of a T1 FS $A$ is given by Carlsson and Fullér [10] as "*the expected value of the squared deviations between the arithmetic mean and the endpoints of its level sets*", i.e.,

$$v(A) = \int_0^1 \alpha\left(\left[\frac{a_1(\alpha)+a_2(\alpha)}{2} - a_1(\alpha)\right]^2 + \left[\frac{a_1(\alpha)+a_2(\alpha)}{2} - a_2(\alpha)\right]^2\right) d\alpha$$
$$= \frac{1}{2}\int_0^1 \alpha[a_2(\alpha) - a_1(\alpha)]^2 d\alpha, \tag{32}$$

where $[a_2(\alpha), a_1(\alpha)]$ is an $\alpha$-cut [36] on $A$.

Lee and Li [39] defined the variance of a T1 FS based on the probability measures of fuzzy events. When the fuzzy events are uniformly distributed, their definition becomes[5]

$$v(A) = \frac{\sum_{i=1}^{N}[x_i - c(A)]^2 \mu_A(x_i)}{\sum_{i=1}^{N}\mu_A(x_i)}, \tag{33}$$

where $c(A)$ is defined in (7).

One way to define the variance $V_{\tilde{A}}$ of an IT2 FS $\widetilde{A}$ is to find the union of the variances of all its embedded T1 FSs $A_e$, i.e.,

$$V_{\tilde{A}} \equiv \bigcup_{\forall A_e} v(A_e) = \bigcup_{\forall A_e}\left[\frac{\sum_{i=1}^{N}[x_i - c(A_e)]^2 \mu_{A_e}(x_i)}{\sum_{i=1}^{N}\mu_{A_e}(x_i)}\right]. \tag{34}$$

There does not seem to be any practical way to compute $V_{\tilde{A}}$ except to compute the variances of all $A_e$ and to then find their union. Because there are an uncountable number of $A_e$, this method is not possible. The following *relative variance* of $A_e$ to $\widetilde{A}$ is introduced, after which it is used to define the variance of $\widetilde{A}$.

[5] In [67] a different form of (33) is used, where the denominator is $N$ instead of $\sum_{i=1}^{N}\mu_A(x_i)$; however, we prefer the definition in (33) because of its analogy to the variance definition in probability theory.

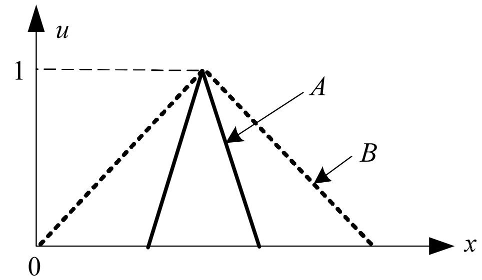


Fig. 4. Illustration of the variance of T1 FSs.

**Definition 5.** The relative variance of an embedded T1 FS $A_e$ to an IT2 FS $\widetilde{A}$, $v_{\tilde{A}}(A_e)$, is defined as

$$v_{\tilde{A}}(A_e) = \frac{\sum_{i=1}^{N}[x_i - c(\widetilde{A})]^2 \mu_{A_e}(x_i)}{\sum_{i=1}^{N}\mu_{A_e}(x_i)}, \tag{35}$$

where

$$c(\widetilde{A}) = \frac{c_l(\widetilde{A}) + c_r(\widetilde{A})}{2} \tag{36}$$

is the center of the centroid of $\widetilde{A}$, $C_{\tilde{A}}$, that is given in (8).

The difference between (35) and (34) is that in (35) the variance of $A_e$ is evaluated relative to $c(\widetilde{A})$, the center of the centroid of $\widetilde{A}$, whereas in (34) the variance of $A_e$ is evaluated relative to $c(A_e)$, the centroid of $A_e$, and, $c(\widetilde{A})$ is computed one time, whereas $c(A_e)$ has to be computed for each $A_e$.

**Definition 6.** The variance of an IT2 FS $\widetilde{A}$, $V_{\tilde{A}}$, is the union of relative variance of all its embedded T1 FSs $A_e$, i.e.,

$$V_{\tilde{A}} \equiv \bigcup_{\forall A_e} v_{\tilde{A}}(A_e) = [v_l(\widetilde{A}), v_r(\widetilde{A})], \tag{37}$$

where $v_l(\widetilde{A})$ and $v_r(\widetilde{A})$ are the minimum and maximum relative variance of all $A_e$, respectively, i.e.

$$v_l(\widetilde{A}) = \min_{\forall A_e} v_{\tilde{A}}(A_e), \tag{38}$$

$$v_r(\widetilde{A}) = \max_{\forall A_e} v_{\tilde{A}}(A_e). \tag{39}$$

Note the analogy of $v_l(\widetilde{A})$ [$v_r(\widetilde{A})$] to $c_l(\widetilde{A})$ defined in (9) [$c_r(\widetilde{A})$ in (10)], and observe also that (35) is the same as (7) except that $[x_i - c(\widetilde{A})]^2$ in (35) takes the place of $x_i$ in (7). Consequently, the iterative KM Algorithms can also be used to compute $v_l(\widetilde{A})$ and $v_r(\widetilde{A})$; however, $[x_i - c(\widetilde{A})]^2$ $(i = 1, 2, \ldots, N)$ need to be sorted in ascending order before the KM Algorithms can be used. The details will be illustrated in Example 6.

**Definition 7.** The standard deviation of an IT2 FS $\widetilde{A}$, $\mathrm{STD}(\widetilde{A})$, is defined as

$$\mathrm{STD}(\widetilde{A}) = V_{\tilde{A}}^{1/2} = \left[\sqrt{v_l(\widetilde{A})}, \sqrt{v_r(\widetilde{A})}\right]. \tag{40}$$

The relationship between the centroid and standard deviation of $\widetilde{A}$ is shown in Fig. 5. $\sqrt{v_l(\widetilde{A})}$ $\left[\sqrt{v_r(\widetilde{A})}\right]$ is an indicator of the compactness of the most (least) compact embedded T1 FS of $\widetilde{A}$, and $\sqrt{v_r(\widetilde{A})} - \sqrt{v_l(\widetilde{A})}$ is an indicator of the area of the FOU.

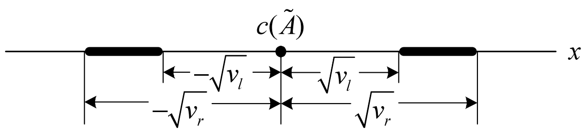


Fig. 5. The standard deviation of $\tilde{A}$.

**Example 6.** [6]Consider the IT2 FS $\tilde{A}$ in Fig. 2d, which is the same as the IT2 FS shown in Fig. 2a. From Example 1 it is known that $c(\tilde{A}) = (c_l(\tilde{A}) + c_r(\tilde{A}))/2 = 3.31$. The original $x_i$ and $y_i \equiv [x_i - c(\tilde{A})]^2$ $(i = 1, \ldots, 8)$ are shown in the first part of Table 3. Before KM Algorithms can be used, $y_i$ need to be sorted in ascending order. The sorted $y_i$, which are called $y'_j$, and the corresponding $x'_j$, $\overline{\mu}_{\tilde{A}}(x'_j)$ and $\underline{\mu}_{\tilde{A}}(x'_j)$ are shown in the second part of Table 3. $V_{\tilde{A}}$ computed by the KM Algorithms is $[0.76, 2.47]$, and consequently, $\text{STD}(\tilde{A}) = [0.87, 1.57]$.

If we map $(y'_j, \underline{\mu}_{\tilde{A}}(x'_j), \overline{\mu}_{\tilde{A}}(x'_j))$ to an FOU as shown in Fig. 6, where the lower membership for $y'_j$ is $\underline{\mu}_{\tilde{A}}(x'_j)$ and the upper membership is $\overline{\mu}_{\tilde{A}}(x'_j)$, then computing $V_{\tilde{A}}$ in (37) is equivalent to computing the centroid of this FOU. The $y'$ domain embedded T1 FSs determining $v_l(\tilde{A})$ and $v_r(\tilde{A})$ are also depicted in Fig. 6.

We can also visualize the corresponding two embedded T1 FSs in the $x$ domain. Because the switch point for $v_l(\tilde{A})$ in the $y'$ domain is $y'_2$, we see from Table 3 that the upper memberships of $x'_1 = 3$ and $x'_2 = 4$ and the lower memberships of all other $x'_j$ should be used to compute $v_l(\tilde{A})$. The corresponding embedded T1 FS is shown in Fig. 2d as the dashed curve. Similarly, because the switch point for $v_r(\tilde{A})$ in the $y'$ domain is $y'_3$, from Table 3 we see that the lower memberships of $x'_1 = 3$, $x'_2 = 4$ and $x'_3 = 2$ and the upper memberships of all other $x'_j$ should be used to compute $v_r(\tilde{A})$. The embedded T1 FS for determining $v_r(\tilde{A})$ is shown in Fig. 2d as the solid curve. Observe that both T1 FSs in the $x$ domain have two switch points.

Note that in this example we plot Fig. 6 only for illustration purpose. In practice, only Table 3 is needed to compute $V_{\tilde{A}}$, and there is no need to visualize the embedded T1 FSs.

### 3.5. Skewness of an IT2 FS

The *skewness* of a T1 FS $A$, $s(A)$, is an indicator of its symmetry. $s(A)$ is smaller than zero when $A$ skews to the right, is larger than zero when $A$ skews to the left, and is equal to zero when $A$ is symmetrical.

**Example 7.** In Fig. 7, $A$ has skewness smaller than zero because it skews to the right, $B$ has skewness larger than zero because it skews to the left, and $C$ has skewness zero because it is symmetrical.

There are a few different definitions of skewness for T1 FSs. Subasic and Nakatsuyama's [61] used

$$s_{\text{SN}}(A) = m_c(A) - m_s(A), \tag{41}$$

where $m_c(A)$ is the center of the core of $A$ and $m_s(A)$ is the center of the support of $A$.

In [7] Bonissone used the following definition:

$$s_{\text{B}}(A) = \sum_{i=1}^{N}[x_i - c(A)]^3 \mu_A(x_i). \tag{42}$$

Since the centroid, variance and skewness of an IT2 FS may be viewed as its first-, second- and third-order moments, respectively, their definitions should be consistent. Consequently, in this paper the following definition is used[7]:

$$s(A) = \frac{\sum_{i=1}^{N}[x_i - c(A)]^3 \mu_A(x_i)}{\sum_{i=1}^{N}\mu_A(x_i)}. \tag{43}$$

[6] Unlike other examples, this example is explained in greater detail because the embedded T1 FSs determining $v_l(\tilde{A})$ and $v_r(\tilde{A})$ have two switch points. This is the first time that the use of the KM Algorithms gives more than one switch point.

[7] In [67] a different form of (43) is used, where the denominator is $N$ instead of $\sum_{i=1}^{N}\mu_A(x_i)$; however, we prefer the definition in (43) because of its analogy to the skewness definition in probability theory.

Table 3
$x_i$ and $y_i \equiv [x_i - c(\widetilde{A})]^2$ $(i = 1, \ldots, 8)$ for IT2 FS $\widetilde{A}$ shown in Fig. 2d

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $x_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $y_i \equiv [x_i - c(\widetilde{A})]^2$ | 10.94 | 5.33 | 1.71 | 0.09 | 0.48 | 2.86 | 7.25 | 13.63 |
| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $y'_j$ | 0.09 | 0.48 | 1.71 | 2.86 | 5.33 | 7.25 | 10.94 | 13.63 |
| $x'_j$ | 3 | 4 | 2 | 5 | 1 | 6 | 0 | 7 |
| $\overline{\mu}_{\widetilde{A}}(x'_j)$ | 1 | 1 | 1 | 0.67 | 0.5 | 0.33 | 0 | 0 |
| $\underline{\mu}_{\widetilde{A}}(x'_j)$ | 0.8 | 0.53 | 0.4 | 0.27 | 0 | 0 | 0 | 0 |

The sorted $y'_j$ and the corresponding $x'_j$, $\overline{\mu}_{\widetilde{A}}(x'_j)$ and $\underline{\mu}_{\widetilde{A}}(x'_j)$ are shown in the second part of the table.
Note that $c(\widetilde{A}) = 3.31$.

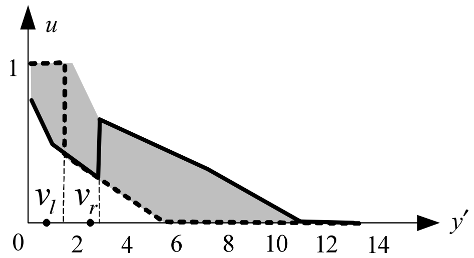


Fig. 6. The $y'$ domain embedded T1 FSs determining $v_l(\widetilde{A})$ (dashed curve) and $v_r(\widetilde{A})$ (solid curve).

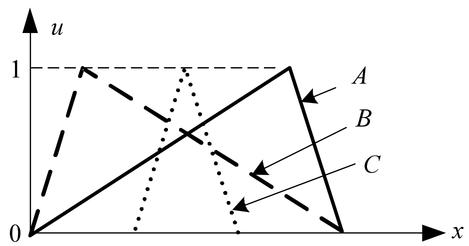


Fig. 7. Illustration of the skewness of T1 FSs.

One way to define the skewness of an IT2 FS $\widetilde{A}$, $S_{\widetilde{A}}$, is to find the union of the skewness of all its embedded T1 FSs $A_e$, i.e.,

$$S_{\widetilde{A}} \equiv \bigcup_{\forall A_e} s(A_e) = \bigcup_{\forall A_e} \left[ \frac{\sum_{i=1}^{N} [x_i - c(A_e)]^3 \mu_{A_e}(x_i)}{\sum_{i=1}^{N} \mu_{A_e}(x_i)} \right]. \tag{44}$$

Again, there does not seem to be any practical way to compute $S_{\widetilde{A}}$ except to compute the skewness of all $A_e$ and to then find their union. Because there are an uncountable number of $A_e$, this method is also not possible. The following *relative skewness* of $A_e$ to $\widetilde{A}$ is introduced, after which it is used to define the skewness of $\widetilde{A}$.

**Definition 8.** The relative skewness of an embedded T1 FS $A_e$ to an IT2 FS $\widetilde{A}$, $s_{\widetilde{A}}(A_e)$, is defined as

$$s_{\widetilde{A}}(A_e) = \frac{\sum_{i=1}^{N} [x_i - c(\widetilde{A})]^3 \mu_{A_e}(x_i)}{\sum_{i=1}^{N} \mu_{A_e}(x_i)}, \tag{45}$$

where $c(\widetilde{A})$ is the center of the centroid of $\widetilde{A}$ [see (36)].

The difference between (45) and (44) is that in (45) the skewness of $A_e$ is evaluated relative to $c(\widetilde{A})$, the center of the centroid of $\widetilde{A}$, whereas in (44) the skewness of $A_e$ is evaluated relative to $c(A_e)$, the centroid of $A_e$, and, $c(\widetilde{A})$ is computed one time, whereas $c(A_e)$ has to be computed for each $A_e$.

**Definition 9.** The skewness of an IT2 FS $\widetilde{A}$, $S_{\tilde{A}}$, is the union of relative skewness of all its embedded T1 FSs $A_e$, i.e.,

$$S_{\tilde{A}} \equiv \bigcup_{\forall A_e} s_{\tilde{A}}(A_e) = [s_l(\widetilde{A}), s_r(\widetilde{A})], \tag{46}$$

where $s_l(\widetilde{A})$ and $s_r(\widetilde{A})$ are the minimum and maximum relative skewness of all $A_e$, respectively, i.e.

$$s_l(\widetilde{A}) = \min_{\forall A_e} s_{\tilde{A}}(A_e), \tag{47}$$

$$s_r(\widetilde{A}) = \max_{\forall A_e} s_{\tilde{A}}(A_e). \tag{48}$$

Observe that (45) is the same as (7) except that $[x_i - c(\widetilde{A})]^3$ in (45) takes the position of $x_i$ in (7). Consequently, the iterative KM Algorithms can also be used to compute $s_l(\widetilde{A})$ and $s_r(\widetilde{A})$.

**Example 8.** Consider the IT2 FS $\widetilde{A}$ in Fig. 2e, which is the same as the IT2 FS shown in Fig. 2a. From Example 6 it is known that $c(\widetilde{A}) = 3.31$. $S_{\tilde{A}}$ computed by the KM Algorithms is $[-2.23, 3.28]$.

Observe from Fig. 2e that the embedded T1 FS determining $s_l(\widetilde{A})$ switches from the UMF of $\widetilde{A}$ to the LMF as $x$ increases, whereas the embedded T1 FS determining $s_r(\widetilde{A})$ switches from the LMF to the UMF as $x$ increases.

## 4. Conclusions

In this paper, five uncertainty measures for IT2 FSs-centroid, cardinality, fuzziness (entropy), variance and skewness – have been introduced. The latter four were newly defined. All measures used the Mendel–John Representation Theorem for IT2 FSs. Formulas for computing these measures were also obtained. Interestingly, observe from Fig. 2 that different embedded T1 FSs are used to compute each of these measures, and the LMF and UMF, which completely determine the FOU, are only used in computing the cardinality of an IT2 FS.

These measures can be used to extend the *principles of uncertainty* [33,25] from T1 FSs to IT2 FSs, and this remains to be done. Finally, the centroid and cardinality have already been used to compute the *similarity* of two IT2 FSs in [69].

## Appendix A. The KM Algorithms

The **KM Algorithm for computing** $c_l(\widetilde{A})$ is [28,46]

(1) Initialize $\theta_i$ by setting

$$\theta_i = [\underline{\mu}_{\tilde{A}}(x_i) + \overline{\mu}_{\tilde{A}}(x_i)]/2, \quad i = 1, 2, \ldots, N \tag{A.1}$$

and then compute

$$c_l' = \frac{\sum_{i=1}^{N} x_i \theta_i}{\sum_{i=1}^{N} \theta_i}. \tag{A.2}$$

(2) Find $k$ $(1 \leqslant k \leqslant N-1)$ such that

$$x_k \leqslant c_l' \leqslant x_{k+1}. \tag{A.3}$$

(3) Set

$$\theta_i = \begin{cases} \overline{\mu}_{\tilde{A}}(x_i), & i \leqslant k, \\ \underline{\mu}_{\tilde{A}}(x_i), & i > k, \end{cases} \tag{A.4}$$

and compute

$$c_l'' = \frac{\sum_{i=1}^{N} x_i\theta_i}{\sum_{i=1}^{N}\theta_i}. \tag{A.5}$$

(4) Check if $c_l'' = c_l'$. If yes, stop, set $c_l(\widetilde{A}) = c_l'$ and call $k$ $L$. If no, go to Step 5.
(5) Set $c_l' = c_l''$ and go to Step 2.

The **KM Algorithm for computing** $c_r(\widetilde{A})$ is [28,46]

(1) Initialize $\theta_i$ as in (A.1) and then compute the right hand side of (A.2), calling it $c_r'$.
(2) Find $k$ ($1 \leqslant k \leqslant N-1$) such that

$$x_k \leqslant c_r' \leqslant x_{k+1}. \tag{A.6}$$

(3) Set

$$\theta_i = \begin{cases} \underline{\mu}_{\widetilde{A}}(x_i), & i \leqslant k, \\ \overline{\mu}_{\widetilde{A}}(x_i), & i > k, \end{cases} \tag{A.7}$$

and compute the right hand side of (A.5), calling it $c_r''$.
(4) Check if $c_r'' = c_r'$. If yes, stop, set $c_r(\widetilde{A}) = c_r'$ and call $k$ $R$. If no, go to Step 5.
(5) Set $c_r' = c_r''$ and go to Step 2.

An enhanced version of the KM algorithms has been proposed in [68]. On average it can save about 39% of the computation time.

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