Type-2 FLS Modeling Capability Analysis

Dongrui Wu
Department of Electrical and
Computer Engineering
National University of Singapore
Singapore 117576
E-mail: g0305786@nus.edu.sg

Woei Wan Tan
Department of Electrical and
Computer Engineering
National University of Singapore
Singapore 117576
E-mail: eletanww@nus.edu.sg

***Abstract*— There has been an increasing amount of research on type-2 fuzzy logic systems (FLSs) recently. The interest is fueled by results demonstrating that type-2 fuzzy sets offer a framework for effectively solving problems where uncertainties are present. A concept, known as the footprint of uncertainty (FOU), is mainly responsible for the improved modeling capability of type-2 FLSs. This paper aims at providing insight into how the extra mathematical dimension provided by the FOU differentiates type-2 FLSs from type-1 FLSs. Since the input-output relationships of both types of FLS are fixed once the parameters are selected, the analysis is performed by finding a set of *equivalent type-1 sets (ET1Ss)* that re-produces the input-output map of a type-2 FLS. Results are presented to demonstrate that a type-2 fuzzy system is able to model more complex input-output relationship because the ET1S changes as the input varies. The technique for converting a type-2 fuzzy set into a group of type-1 sets is also useful as it provides a framework for extending the entire wealth of type-1 fuzzy control/identification/design/analysis techniques to type-2 systems.**

I. Introduction

Type-2 fuzzy sets was introduced by Zadeh in 1975 [1] as an extension of the type-1 set. A type-2 fuzzy set is characterised by a concept called footprint of uncertainty (FOU). Consequently, the membership grade of each element in a type-2 fuzzy set is a fuzzy set in $[0, 1]$, unlike a type-1 set where the membership grade is a crisp number in $[0, 1]$. Fuzzy logic systems (FLSs) constructed using type-2 fuzzy sets are type-2 FLSs to distinguish them from the traditional type-1 FLSs. Fig. 1 shows the schematic diagram of a type-2 FLS. A type-reducer is needed to convert the type-2 fuzzy output sets into type-1 sets before they are processed by the defuzzifier to give a crisp output. Since type-2 FLSs provide an extra mathematical dimension compared with type-1 FLSs, they are very useful in circumstances where it is difficult to determine an exact membership grade for a fuzzy set. Hence, they can be used to handle more system uncertainties and have the potential to outperform their type-1 counterparts. To date, type-2 FLSs have been used successfully in decision making [2], control of mobile robots [3], preprocessing of data [4], noise cancellation [5], time-series forecasting [6], survey processing [6], [7], nonlinear system identification [8] and control [9], [10], etc.

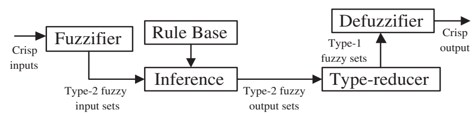


Fig. 1. A type-2 fuzzy logic system

Though the FOU provides type-2 FLSs with the potential to outperform type-1 FLSs, how to choose the best FOU is still an open question. Several researchers have demonstrated that genetic algorithms can be used to evolve the FOU [8], [9], [11], [12]. However, there are no guidelines on how to decide the FOU theoretically. It is also unclear how the FOU enables type-2 FLSs to differentiate themselves from their type-1 counterparts. The purpose of this paper is to study the role of the FOU. Results reported herein may help to explain why type-2 FLSs are able to model more complex input-output relationships.

The rest of the paper is organized as follows: Section II introduces the concept of equivalent type-1 sets and describes the strategy for finding the equivalent sets that will re-produce the input-output map of a particular type-2 FLS. The methodology for identifying ET1Ss is then used to study the characteristics of a type-2 set obtained by blurring a triangular type-1 set in Section III. Section IV discusses the implications of the results. Finally, conclusions are drawn in Section V.

II. Equivalent Type-1 Fuzzy Sets

This paper aims at understanding how the extra degree of freedom provided by the FOU enables type-2 FLSs to produce more complex input-output maps. The key idea is that a type-2 set can be reduced to a group of type-1 sets without affecting the output of a type-2 FLS since the input-output relationships of both type-1 and type-2 FLS are fixed once the parameters are selected. By analyzing the characteristics of an *equivalent type-1 set* (ET1S), conclusions about the contributions of the FOU can be drawn. Before presenting the method to identify the ET1Ss for a given type-2 FLS, the proposed concept needs to be formally introduced.

> By definition, *equivalent type-1 sets* is the collection of type-1 sets that can be used in place of the FOUs in a type-2 FLS.

A. Rationale for ET1Ss

The focus of this subsection is the rationale behind ET1Ss. First, the type-2 FLS studied is described. As the objective is to provide insights into how a type-2 FLS differs from its type-1 counterpart, a common basis for comparison is needed. Hence, the strategy adopted is to start with a baseline type-1 FLS. The fuzzy Proportional plus Integral (PI) controller [13] is selected because fuzzy control is one of the most common applications of fuzzy theory. It has two inputs signals, the feedback error (e) and its rate of change ($\dot{e}$). The output signal is the rate of change in the control action ($\dot{u}$). For simplicity, each input domain is characterized by 3 type-1 fuzzy sets. The rule base is shown in Table I. When the "Product-Sum-Gravity Method" is used to implement the inference engine, a PI controller $u(t) = K_P e(t) + K_I \int_0^t e(\tau)d\tau$ can be realized by a FLS if the consequences in Table I are defined as :

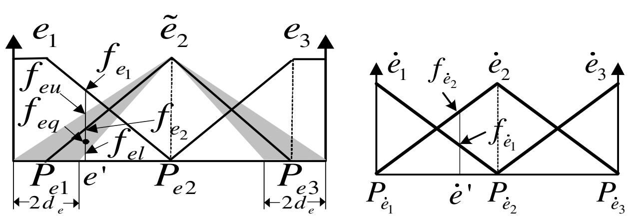
$$\dot{u}_{ij} = K_I \cdot P_{e_i} + K_P \cdot P_{\dot{e}_j} \qquad i, j = 1, 2, 3 \tag{1}$$

where P_{e_i} is the apex of the membership function (MF) e_i, $P_{\dot{e}_j}$ is the apex of MF $\dot{e}_j$, as labelled in Fig. 2.

TABLE I

RULE BASE OF THE FLSS

$e \setminus \dot{e}$	$\dot{e}_1$	$\dot{e}_2$	$\dot{e}_3$
e_1	$\dot{u}_{11}$	$\dot{u}_{12}$	$\dot{u}_{13}$
e_2	$\dot{u}_{21}$	$\dot{u}_{22}$	$\dot{u}_{23}$
e_3	$\dot{u}_{31}$	$\dot{u}_{32}$	$\dot{u}_{33}$

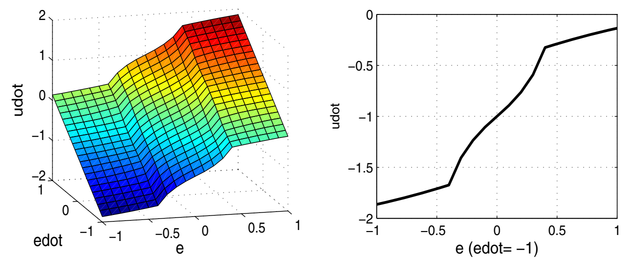


(a) MFs of input e (b) MFs of input $\dot{e}$

Fig. 2. MFs of the two FLSs

The type-2 FLS studied is one with a simple but effective architecture [10]. Fig. 2 shows the membership functions of the type-2 FLS. The input signals, output signal and the rule base are the same as the baseline type-1 FLS. The only difference is the center fuzzy set in the error domain, which has been changed into a type-2 fuzzy set. $\tilde{e}_2$ is obtained by blurring the type-1 FLS, whose MFs are shown as the dark thick lines. The blurred area, referred to as the footprint of uncertainty (FOU), represents the uncertainties in the shape and position of the type-1 fuzzy set. It is bounded by an upper MF and a lower MF, both of which are type-1 MFs. The type-2 fuzzy set used is an interval one i.e. each point in the FOU has unity secondary membership grade.

Fig. 3(a) shows a sample input-output map of the type-2 FLS. Clearly, it is non-linear. Since the baseline type-1 FLS is equivalent to a linear PI controller, it may be concluded from the plot in Fig. 3(a) that a more complex relationship can be modeled by simply changing one of the fuzzy sets from type-1 to type-2. Although the input-output map of the type-2 FLS may be relatively more complex, the output corresponding to a particular set of inputs is still fixed once the system parameters are selected. As a type-1 FLS has the same property, the implication is that the interval firing strength of the type-2 set corresponding to a particular input-output pair can effectively be replaced by a crisp value without affecting the system output. The task of finding the equivalent type-1 sets can be achieved by first making a vertical cut to obtain a slice where all points have the same $\dot{e}$ value. That is, for a particular input $\dot{e}$, a curve representing the relationship between the output $\dot{u}$ and the input e can be plotted (See Fig. 3(b) when $\dot{e} = -1$). Each slice is then replicated by replacing the type-2 set, $\tilde{e}_2$, with an ET1S. Since the surface corresponding to the type-2 FLS is more complex, the shape of the slice may change as $\dot{e}$ is varied so different ET1S may be needed to re-produce the curve corresponding to different $\dot{e}$. By considering all $\dot{e}$ within the universe of discourse $[p_{\dot{e}_1}, p_{\dot{e}_3}]$, the collection of ET1Ss that duplicates all the slices and therefore the input-output map can be found. The following subsection formally delineates a procedure for identifying the ET1Ss.



(a) Control surface of a type-2 FLS (b) A slice at $\dot{e} = -1$

Fig. 3. Illustration of the control surface and a slice of it

B. *Method for identifying ET1Ss*

Consider the input pair (e', $\dot{e}'$). Suppose the firing strength of the type-2 MF $\tilde{e}_2$ corresponding to e' ($f_{\tilde{e}_2}$) is the interval set $[f_{e_l},\ f_{e_u}]$, where f_{e_l} and f_{e_u} are the points of intersection on the lower and upper MFs of the type-2 set $\tilde{e}_2$ as labelled in Fig. 2(a). Assume the firing strengths of the type-1 sets e_1, e_3, $\dot{e}_1$, $\dot{e}_2$ and $\dot{e}_3$ are f_{e_1}, f_{e_3}, $f_{\dot{e}_1}$, $f_{\dot{e}_2}$ and $f_{\dot{e}_3}$, respectively. Then, the firing strengths of the rules are :

$$R^{ij} : \begin{cases} f_{e_i} \times f_{\dot{e}_j} \rightarrow \dot{u}_{ij} & i = 1, 3 \\ f_{\tilde{e}_i} \star f_{\dot{e}_j} = [f_{e_l} \times f_{\dot{e}_j},\ f_{e_u} \times f_{\dot{e}_j}] \rightarrow \dot{u}_{ij} & i = 2 \end{cases}$$

where $j = 1, 2, 3$. The crisp output, $\dot{u}'$, of the type-2 FLS corresponding to the input (e', $\dot{e}'$) is obtained using sum-product inference, center-of-sets type-reduction and height defuzzification.

The strategy for identifying ET1Ss is based on the principle of reducing the interval firing set, $f_{\tilde{e}_2}$, to a single value without affecting the crisp output $\dot{u}'$ of the type-2 FLS. Let the *equivalent type-1 membership grade*, f_{eq}, be a point on the MF of the ET1S corresponding to $\dot{e} = \dot{e}'$ (Refer to Fig. 2(a)). When the type-2 fuzzy set is replaced by its ET1S, the type-2 FLS is reduced to an *equivalent type-1 FLS*. Assuming height defuzzification, the crisp output $\dot{u}_{eq}$ of the *equivalent type-1 FLS* is the mathematical expression labelled as Equation (2).

$$\dot{u}_{eq} = \frac{f_{e_1} f_{\dot{e}_1} \dot{u}_{11} + f_{e_1} f_{\dot{e}_2} \dot{u}_{12} + f_{e_1} f_{\dot{e}_3} \dot{u}_{13} + f_{eq} f_{\dot{e}_1} \dot{u}_{21} + f_{eq} f_{\dot{e}_2} \dot{u}_{22} + f_{eq} f_{\dot{e}_3} \dot{u}_{23} + f_{e_3} f_{\dot{e}_1} \dot{u}_{31} + f_{e_3} f_{\dot{e}_2} \dot{u}_{32} + f_{e_3} f_{\dot{e}_3} \dot{u}_{33}}{f_{e_1} f_{\dot{e}_1} + f_{e_1} f_{\dot{e}_2} + f_{e_1} f_{\dot{e}_3} + f_{eq} f_{\dot{e}_1} + f_{eq} f_{\dot{e}_2} + f_{eq} f_{\dot{e}_3} + f_{e_3} f_{\dot{e}_1} + f_{e_3} f_{\dot{e}_2} + f_{e_3} f_{\dot{e}_3}} \quad (2)$$

$$f_{eq} = \frac{\dot{u}'(f_{e_1} + f_{e_3})(f_{\dot{e}_1} + f_{\dot{e}_2} + f_{\dot{e}_3}) - f_{e_1}(f_{\dot{e}_1} \dot{u}_{11} + f_{\dot{e}_2} \dot{u}_{12} + f_{\dot{e}_3} \dot{u}_{13}) - f_{e_3}(f_{\dot{e}_1} \dot{u}_{31} + f_{\dot{e}_2} \dot{u}_{32} + f_{\dot{e}_3} \dot{u}_{33})}{f_{\dot{e}_1} \dot{u}_{21} + f_{\dot{e}_2} \dot{u}_{22} + f_{\dot{e}_3} \dot{u}_{23} - \dot{u}'(f_{\dot{e}_1} + f_{\dot{e}_2} + f_{\dot{e}_3})} \quad (3)$$

Since the output should not be affected when a type-2 FLS is switched to its ET1S, f_{eq} must be selected to reproduce $\dot{u}'$ i.e. $\dot{u}_{eq} = \dot{u}'$. Consequently, the mathematical expression for calculating the appropriate f_{eq} can be derived by substituting $\dot{u}_{eq}$ by $\dot{u}'$ in Equation (2) and then re-arranging. The complete group of ET1Ss can be then identified by discretizing the input domains and applying Equation (3) repeatedly. In summary, the procedure for finding ET1Ss of a two-inputs single output Type-2 FLS is as follows :-

1) Discretize the $\dot{e}$ domain into m points $[\dot{e}'_1, \dot{e}'_2, \ldots, \dot{e}'_m]$
2) Discretize the FOU of the type-2 set $\tilde{e}_2$, bounded by $(P_{e_1} - d_e)$ and $(P_{e_3} + d_e)$ (Refer to Fig. 2(a)), into n points.
3) Select an element in $[\dot{e}'_1, \dot{e}'_2, \ldots, \dot{e}'_m]$. Use Equation (3) to calculate the n equivalent type-1 membership grades $f_{eq,i} (i = 1, \ldots, n)$ such that the slice of the input-output map corresponding to the selected $\dot{e}'$ remains unchanged. By joining the n equivalent type-1 membership grades, the ET1S corresponding to a particular $\dot{e}'$ can be found.
4) Repeat Step (3) for the remaining $(m-1)$ elements in $[\dot{e}'_1, \dot{e}'_2, \ldots, \dot{e}'_m]$ as the ET1Ss may differ from each other as $\dot{e}'$ changes.

Here it is assumed only one MF of the type-2 FLS is type-2. When there are k $(k > 1)$ type-2 sets, one can first replace $k-1$ type-2 sets by $k-1$ embedded type-1 sets [6] and then find ET1Ss to replace the last type-2 set [14].

C. Uniqueness of ET1S corresponding to each $\dot{e}'$

This subsection provides proof that an unique *equivalent type-1 membership grade* f_{eq}, and therefore a unique ET1S, exists when the remaining firing strengths in a FLS are fixed.

First, assume $e' \in [P_{e_1} - d_e, P_{e_2}]$ and $\dot{e}' \in [P_{\dot{e}_1}, P_{\dot{e}_2}]$. In this case, only the MFs e_1, $\tilde{e}_2$, $\dot{e}_1$ and $\dot{e}_2$ are fired and the firing strengths are f_{e_1}, $f_{\tilde{e}_2}$, $f_{\dot{e}_1}$ and $f_{\dot{e}_2}$, respectively. Equation (2) can, therefore, be simplified to Equation (4).

$$\begin{aligned}
\dot{u}_{eq} &= \frac{f_{e_1} f_{\dot{e}_1} \dot{u}_{11} + f_{e_1} f_{\dot{e}_2} \dot{u}_{12} + f_{eq} f_{\dot{e}_1} \dot{u}_{21} + f_{eq} f_{\dot{e}_2} \dot{u}_{22}}{f_{e_1} f_{\dot{e}_1} + f_{e_1} f_{\dot{e}_2} + f_{eq} f_{\dot{e}_1} + f_{eq} f_{\dot{e}_2}} \\
&= \frac{(f_{e_1} f_{\dot{e}_1} \dot{u}_{11} + f_{e_1} f_{\dot{e}_2} \dot{u}_{12}) + f_{eq}(f_{\dot{e}_1} \dot{u}_{21} + f_{\dot{e}_2} \dot{u}_{22})}{(f_{e_1} f_{\dot{e}_1} + f_{e_1} f_{\dot{e}_2}) + f_{eq}(f_{\dot{e}_1} + f_{\dot{e}_2})} \\
&= \frac{(f_{e_1} f_{\dot{e}_1} \dot{u}_{11} + f_{e_1} f_{\dot{e}_2} \dot{u}_{12}) - \frac{f_{\dot{e}_1} \dot{u}_{21} + f_{\dot{e}_2} \dot{u}_{22}}{f_{\dot{e}_1} + f_{\dot{e}_2}}(f_{e_1} f_{\dot{e}_1} + f_{e_1} f_{\dot{e}_2})}{(f_{e_1} f_{\dot{e}_1} + f_{e_1} f_{\dot{e}_2}) + f_{eq}(f_{\dot{e}_1} + f_{\dot{e}_2})} \\
&\quad + \frac{f_{\dot{e}_1} \dot{u}_{21} + f_{\dot{e}_2} \dot{u}_{22}}{f_{\dot{e}_1} + f_{\dot{e}_2}} \\
&= \frac{1}{(f_{e_1} + f_{eq})(f_{\dot{e}_1} + f_{\dot{e}_2})^2} \Big[f_{e_1} f_{\dot{e}_1}^2 (\dot{u}_{11} - \dot{u}_{21}) \\
&\quad + f_{e_1} f_{\dot{e}_1} f_{\dot{e}_2} (\dot{u}_{11} + \dot{u}_{12} - \dot{u}_{21} - \dot{u}_{22}) \\
&\quad + f_{e_1} f_{\dot{e}_2}^2 (\dot{u}_{12} - \dot{u}_{22}) \Big] + \frac{f_{\dot{e}_1} \dot{u}_{21} + f_{\dot{e}_2} \dot{u}_{22}}{f_{\dot{e}_1} f_{\dot{e}_2}}
\end{aligned} \quad (4)$$

Substituting Equation (1) into Equation (4) :

$$\dot{u}_{eq} = \frac{f_{\dot{e}_1} \dot{u}_{21} + f_{\dot{e}_2} \dot{u}_{22}}{f_{\dot{e}_1} + f_{\dot{e}_2}} + \frac{K_I f_{e_1}}{f_{e_1} + f_{eq}} (P_{e_1} - P_{e_2}) \quad (5)$$

The first term on the right hand side of Equation (5) is a constant. Since f_{e_1}, f_{eq}, $f_{\dot{e}_1}$ and $f_{\dot{e}_2}$ are firing strengths, they should satisfy the following relations:

$$\begin{aligned} f_{e_1} \geq 0, \quad f_{eq} \geq 0, \quad f_{e_1} + f_{eq} > 0 \\ f_{\dot{e}_1} \geq 0, \quad f_{\dot{e}_2} \geq 0, \quad f_{\dot{e}_1} + f_{\dot{e}_2} > 0 \end{aligned}$$

From Fig. 2 we have

$$P_{e_1} - P_{e_2} < 0$$

Hence, Equation (5) can be further simplified as :

$$\dot{u}_{eq} = c + \frac{K}{f_{e_1} + f_{eq}} \quad (6)$$

where

$$c = \frac{f_{\dot{e}_1} \dot{u}_{21} + f_{\dot{e}_2} \dot{u}_{22}}{f_{\dot{e}_1} + f_{\dot{e}_2}} \quad (7)$$

is a constant and

$$K = K_I f_{e_1} (P_{e_1} - P_{e_2}) \leq 0 \quad (8)$$

Since $K = 0$ if $f_{e_1} = 0$, $\dot{u}_{eq}$ does not change with f_{eq} when $e' = P_{e_2}$. Otherwise, $K < 0$ and $\dot{u}_{eq}$ will increase when f_{eq} increases. The analysis indicates that $\dot{u}_{eq}$ is monotonic to f_{eq} when all other firing strengths are fixed and $e' \neq P_{e_2}$. Fig. 4 shows the relationships between $\dot{u}_{eq}$ and f_{eq} when $P_{e_1} = P_{\dot{e}_1} = -1$, $P_{e_2} = P_{\dot{e}_2} = 0$, $K_P = 1$, $K_I = 1$, $e' \in \{-0.5, 0\}$ and $\dot{e}' = -0.5$. They coincide with the analysis.

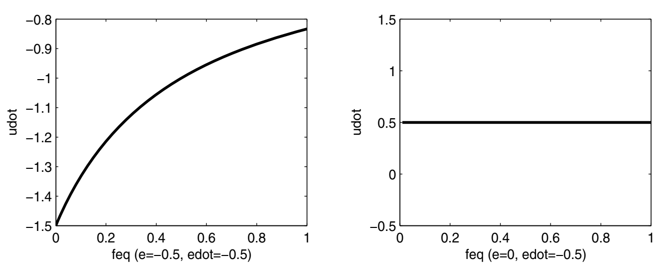


Fig. 4. The monotonicity between $\dot{u}_{eq}$ and f_{eq}

When the inputs are in other regions of the input domain, similar calculations are performed and the following conclusions can be drawn:

$\dot{u}_{eq} \uparrow$ *as* $f_{eq} \uparrow$, *when* $e' < P_{e_2}$ *and* $\dot{e}' \leq P_{\dot{e}_2}$;
$\dot{u}_{eq} \uparrow$ *as* $f_{eq} \uparrow$, *when* $e' < P_{e_2}$ *and* $\dot{e}' \geq P_{\dot{e}_2}$;
$\dot{u}_{eq} \downarrow$ *as* $f_{eq} \uparrow$, *when* $e' > P_{e_2}$ *and* $\dot{e}' \leq P_{\dot{e}_2}$;
$\dot{u}_{eq} \downarrow$ *as* $f_{eq} \uparrow$, *when* $e' > P_{e_2}$ *and* $\dot{e}' \geq P_{\dot{e}_2}$.

More concisely,

$\dot{u}_{eq} \uparrow$ *as* $f_{eq} \uparrow$, *when* $e' < P_{e_2}$;
$\dot{u}_{eq} \downarrow$ *as* $f_{eq} \uparrow$, *when* $e' > P_{e_2}$.

As the relationship between $\dot{u}_{eq}$ and f_{eq} is monotonic when $K \neq 0$, it may be concluded that there is a 1-1 mapping between f_{eq} and $\dot{u}_{eq}$ provided that $e' \neq P_{e_2}$. That is, the f_{eq} calculated using Equation (3) is unique. Consequently, the ET1S is also unique when $\dot{e}'$ is fixed.

III. Simulation Results

The simulation results presented in this section were obtained when the parameters defined in Fig. 2 assume the following values :

$$P_{e_1} = -1, \quad P_{e_2} = 0, \quad P_{e_3} = 1$$
$$P_{\dot{e}_1} = -1, \quad P_{\dot{e}_2} = 0, \quad P_{\dot{e}_3} = 1$$

The parameters of the PI controller are :

$$K_P = 1, \quad K_I = \{0.2,\ 1,\ 2\}$$

When K_I changes, the corresponding rule bases are shown in Table II–IV.

TABLE II
Rule base of the FLSs when $K_I = 0.2$

$e \setminus \dot{e}$	$\dot{e}_1$	$\dot{e}_2$	$\dot{e}_3$
e_1	-1.2	-0.2	0.8
e_2	-1	0	1
e_3	-0.8	0.2	1.2

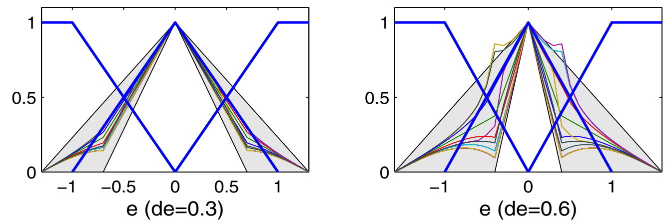
TABLE III
Rule base of the FLSs when $K_I = 1$

$e \setminus \dot{e}$	$\dot{e}_1$	$\dot{e}_2$	$\dot{e}_3$
e_1	-2	-1	0
e_2	-1	0	1
e_3	0	1	2

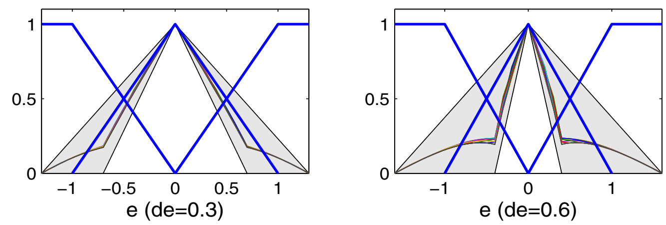
TABLE IV
Rule base of the FLSs when $K_I = 2$

$e \setminus \dot{e}$	$\dot{e}_1$	$\dot{e}_2$	$\dot{e}_3$
e_1	-3	-2	-1
e_2	-1	0	1
e_3	1	2	3

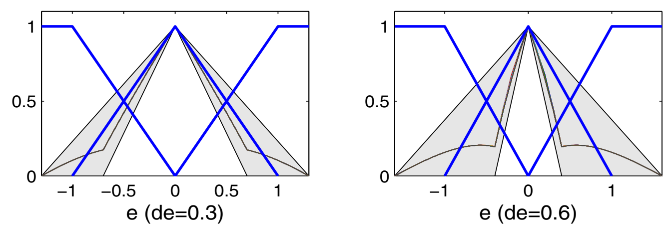
The ET1Ss of $\tilde{e}_2$ may be calculated using the procedure described in Section II-B. By discretizing the domain of e into $\frac{2(1+d_e)}{0.1} + 1$ points and the domain of $\dot{e}$ into 21 points (i.e. the distance between successive discrete points is 0.1), the ET1Ss when the parameter defining the width of the FOU (d_e) is 0.3 and 0.6 respectively were found and shown in Fig. 5. The input-output maps of the resulting FLSs are shown in Fig. 6. Note that the Karnik-Mendel iterative type-reduction procedure [6] was used to produce the type-2 FLS mapping.



(a) $K_I = 0.2$

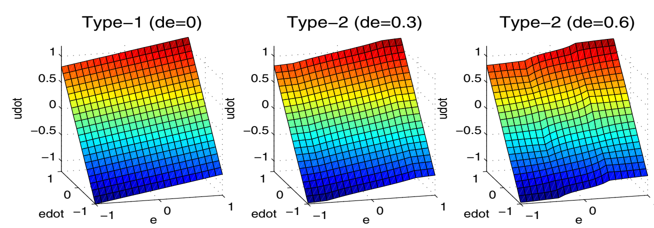


(b) $K_I = 1$

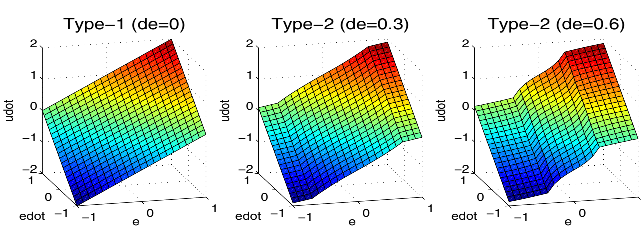


(c) $K_I = 2$

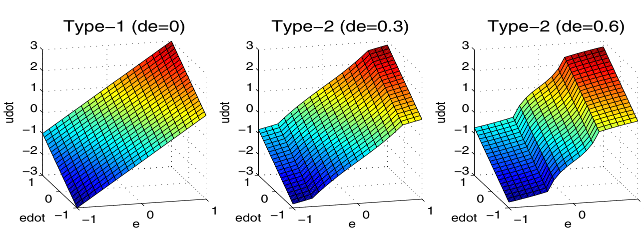
Fig. 5. ET1Ss of $\tilde{e}_2$ when K_I changes



(a) $K_I = 0.2$



(b) $K_I = 1$



(c) $K_I = 2$

Fig. 6. Control surfaces of the FLSs when K_I changes

IV. Analysis and Discussions

Several patterns can be found from Fig. 5 and Fig. 6 :

(1) *A larger FOU gives rise to a more complex input-output relationship.* This observation coincides with intuitions. When the FOU is bigger, the difference between f_{e_u} and f_{e_l} is bigger, which may result in more diverse output and hence more complex control surface.

(2) *As the FOU of a type-2 set grows, the ET1Ss become more diverse.* This characteristics is consistent with observation (1). An ET1S at $\dot{e}'$ is a curve that can be used to replace the type-2 set when $\dot{e} = \dot{e}'$ without changing the output. The output is actually a slice of the control surface at $\dot{e} = \dot{e}'$. Since the control surface is more complex when the FOU is bigger, the difference between different ET1Ss becomes more obvious.

(3) *When the MFs of the type-2 FLS is symmetric, all the ET1Ss as a whole is symmetric. However, a particular ET1S may not be symmetric.* As shown in Fig. 7, the ET1S at $\dot{e} = 0$ is symmetric. The ET1S when $\dot{e} = -0.2$ is symmetric to the one when $\dot{e} = 0.2$. When the type-2 FLS itself is not symmetric, the ET1Ss will not be symmetric.

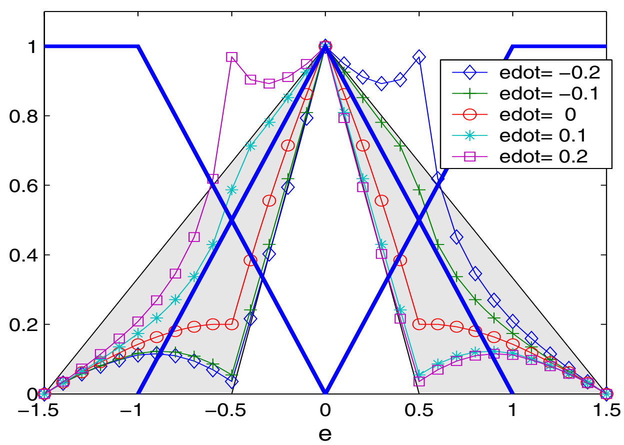
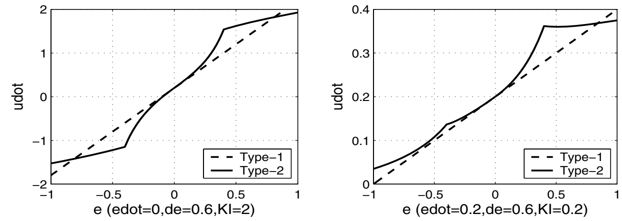


Fig. 7. Illustration of symmetry. $K_P = 1$, $K_I = 0.11$ and $d_e = 0.5$

(4) *The input-output maps of the resulting type-2 FLSs are nonlinear and more complex.* This is the direct result of the shape of the ET1Ss and can be analyzed using Equation (2). The first derivative of $\dot{u}_{eq}$ (slope of $\dot{u}_{eq}$) wrt f_{eq} is :

$$\ddot{u}_{eq} = K_I[f_{e_1}(P_{e_2} - P_{e_1}) + f_{e_3}(P_{e_3} - P_{e_2})] \cdot \frac{1}{(f_{e_1} + f_{eq} + f_{e_3})^2} \tag{9}$$

Equation (9) indicates that $|\ddot{u}_{eq}|$ will increase when f_{eq} decreases. In other words, a smaller f_{eq} means that the slope of $\dot{u}_{eq}$ is steeper. Fig. 5 shows that the membership grades of the ET1Ss are the same as that of the original type-1 set e_2 when $e = 0$. Thus, the slope of the control surfaces are the same at this point. When e gradually departs from P_{e_2}, the membership grades of the ET1Ss can be bigger or smaller than that of the original type-1 MF, thus the input-output map is more complex. Fig. 8 shows two slices of the control surfaces with different $\dot{e}'$ and K_I. Interestingly, the shapes are similar to the ET1Ss shown in Fig. 5.



(a) $K_I = 2$, $\dot{e}' = 0$ (b) $K_I = 0.2$, $\dot{e}' = 0.2$

Fig. 8. The slope of the control surface

(5) *The ET1Ss are closer together if K_I is larger.* By analyzing Equation (9), it may be concluded that the crisp output of a type-2 FLS, $\dot{u}$, is proportional to K_I. When K_I increases, the first derivative of $\dot{u}$ also increases. For the same change in f_{eq}, the FLS corresponding to $K_I = 2$ gives the maximum output change. From another point of view, the FLS with $K_I = 2$ needs the minimum change in f_{eq} to give the same amount of change in $\dot{u}$. Thus, its ET1Ss are the closest of the three FLSs that were studied.

(6) *The ET1Ss may not lie in the FOU of the corresponding type-2 set. Moreover, the equivalent type-1 membership grades of the ET1Ss may be larger than 1 or smaller than 0.* ET1Ss with some *equivalent type-1 membership grades* that are larger than the upper membership grade are illustrated in Fig. 7. More interesting ET1Ss are presented in Fig. 9, where some of the *equivalent type-1 membership grades* of the ET1Ss are negative. To provide insights into why *equivalent type-1 membership grades* may assume negative values, consider the input pair $e = -0.2$, $\dot{e} = 0.2$. The slice of the input-output map where $\dot{e} = 0.2$ is shown in Fig. 10. The point of interest is labelled by a square. In this case, the firing strengths are :

$$\begin{aligned} f_{e_1} &= 0.2 \\ f_{\tilde{e}_2} &= [0,\ 0.8889] \\ f_{\dot{e}_2} &= 0.8 \\ f_{\dot{e}_3} &= 0.2 \end{aligned}$$

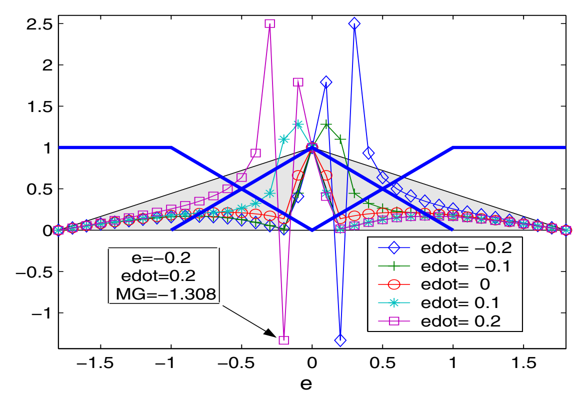


Fig. 9. Example where ET1Ss are not within FOU ($K_I = 0.2$, $d_e = 0.8$)

Thus, the fired rules are :

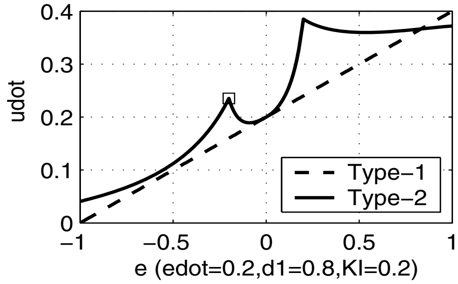


Fig. 10. A slice of the control surface when $K_I = 0.2$, $\dot{e} = 0.2$, $d_e = 0.8$

Rule No:	Firing Strength	$\rightarrow$	Consequent
R^{12} :	0.16	$\rightarrow$	-0.2
R^{13} :	0.04	$\rightarrow$	0.8
R^{22} :	[0, 0.7111]	$\rightarrow$	0
R^{23} :	[0, 0.1778]	$\rightarrow$	1

For the type-2 FLS, the bounds of the type-reduced interval type-1 set obtained using the Karnik-Mendel type-reducer and the resulting crisp output are :

$$\begin{aligned} \dot{u}_l &= \tfrac{-0.16\times 0.2+0.04\times 0.8+0\times 0+0\times 1}{0.04+0.16+0+0} = 0 \\ \dot{u}_r &= \tfrac{-0.16\times 0.2+0.04\times 0.8+0\times 0+0.1778\times 1}{0.04+0.16+0+0.1778} = 0.4722 \\ \dot{u} &= \frac{\dot{u}_l + \dot{u}_r}{2} = 0.2361 \end{aligned} \tag{10}$$

Suppose the *equivalent type-1 membership grade* of the interval firing strength $f_{\tilde{e}_2} = [0,\ 0.8889]$ is f_{eq}. Then, the firing level of rules in the *equivalent type-1 FLS* are :

Rule No:	Firing Strength	$\rightarrow$	Consequence
R^{12} :	0.16	$\rightarrow$	-0.2
R^{13} :	0.04	$\rightarrow$	0.8
R^{22} :	$0.8f_{eq}$	$\rightarrow$	0
R^{23} :	$0.2f_{eq}$	$\rightarrow$	1

The expression governing the output of the *equivalent type-1 FLS* is :

$$\begin{aligned} \dot{u}_{eq} &= \tfrac{0.16\times -0.2+0.04\times 0.8+0.8f_{eq}\times 0+0.2f_{eq}\times 1}{0.16+0.04+0.8f_{eq}+0.2f_{eq}} \\ &= \frac{0.2f_{eq}}{0.2+f_{eq}} \end{aligned} \tag{11}$$

For positive f_{eq}, the output of the *equivalent type-1 FLS*, $\dot{u}_{eq}$ (Equation (11)), will increase and tend towards 0.2 as $f_{eq} \rightarrow +\infty$ i.e.

$$\lim_{f_{eq}\rightarrow +\infty} \frac{0.2f_{eq}}{0.2+f_{eq}} = 0.2$$

Since the maximum $\dot{u}_{eq}$ value is 0.2 if f_{eq} is constrained to be positive, the resulting *equivalent type-1 FLS* will not be able to replicate the crisp output of the type-2 FLS which is 0.2361 (Equation (10)). The only way for the outputs of the *equivalent type-1 FLS* and the type-2 FLS to match is for f_{eq} to take on the negative value -1.3080. This analysis indicates that the extra dimension provided by the FOU enables a type-2 FLS to produce outputs that cannot be achieved by traditional type-1 FLSs with the same number of MFs.

From the above analysis, there are two main differences between type-1 and type-2 FLSs. Firstly, a type-2 fuzzy set can be viewed as a combination of many different ET1Ss. A different ET1S is utilized when the input is changed, thereby providing a type-2 FLS with more degrees of freedom. Secondly, a type-2 fuzzy set may give rise to an *equivalent type-1 membership grade* that is negative or larger than unity. These two characteristics of a type-2 fuzzy set enable a type-2 FLS to model more complex input-output relationships than its type-1 counterpart.

V. Conclusions

In this paper, the role of the FOU in type-2 sets is analyzed by introducing the ET1Ss concept. Using the proposed algorithm for identifying ET1Ss, the characteristics of a type-2 FLS is compared with a type-1 FLS. Results show that the FOU needs to be replicated by different ET1Ss as the inputs are varied. Thus, the type-2 FLS can be viewed as a combination of many different type-1 FLSs. As a type-2 FLS switches between the various type-1 FLSs, it is able to model more complicated input-output maps. The results in this paper may also provide insights on how the FOU can be theoretically selected. That is one of our future research directions.

References

[1] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-1," *Inform. Sci.*, vol. 8, pp. 199–249, 1975.

[2] R. R. Yager, "Fuzzy subsets of type-2 in decisions," *J. Cybern.*, vol. 10, pp. 137–159, 1980.

[3] H. Hagras, "A hierarchical type-2 fuzzy logic control architecture for autonomous mobile robots," *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 4, pp. 524– 539, Aug. 2004.

[4] R. I. John, P. R. Innocent, and M. R. Barnes, "Neuro-fuzzy clustering of radiographic tibia images using type 2 fuzzy sets," *Inform. Sci.*, vol. 125, pp. 65–82, 2000.

[5] O. Castillo and P. Melin, "Adaptive noise cancellation using type-2 fuzzy logic and neural networks," in *Proceedings. FUZZ-IEEE*, vol. 2, 25-29 July 2004, pp. 1093–1098.

[6] J. M. Mendel, *Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. NJ: Prentice-Hall, 2001.

[7] S. Auephanwiriyakul, A. Adrian, and J. M. Keller, "Type-2 fuzzy set analysis in management surveys," in *Proc. FUZZ-IEEE Conf.*, May 2002, pp. 1321–1325.

[8] C. H. Wang, C. S. Cheng, and T. T. Lee, "Dynamical optimal training for interval type-2 fuzzy neural network (T2FNN)," *IEEE Trans. Syst., Man, Cybern. B*, vol. 34, no. 3, pp. 1462– 1477, June 2004.

[9] D. Wu and W. W. Tan, "Genetic learning and performance evaluation of type-2 fuzzy logic controllers," *IEEE Trans. Syst., Man, Cybern. B*, 2004, submitted for publication.

[10] ——, "A simplified architecture for type-2 FLSs and its application to nonlinear control," in *Proceedings. IEEE Conf. On Cybern. and Intell. Syst.*, Singapore, Dec. 2004, pp. 485–490.

[11] P. Innocent, R. John, I. Belton, and D. Finlay, "Type 2 fuzzy representations of lung scans to predict pulmonary emboli," in *Proc. Joint 9th IFSA World Congress and 20th NAFIPS Int. Conf.*, vol. 4, Vancouver, BC, Canada, July 2001, pp. 1902 – 1907.

[12] S. Park and H. Lee-Kwang, "A designing method for type-2 fuzzy logic systems using genetic algorithms," in *Proc. Joint 9th IFSA World Congress and 20th NAFIPS Int. Conf.*, vol. 5, Vancouver, BC, Canada, July 2001, pp. 2567 – 2572.

[13] M. Mizumoto, "Realization of PID controls by fuzzy control methods," *Fuzzy sets and systems*, vol. 70, pp. 171–182, 1995.

[14] W. W. Tan and D. Wu, "Theory of equivalent type-1 fuzzy logic systems (ET1FLSs)," 2005, submitted to IEEE Trans. on Fuzzy Systems.