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Twelve Considerations in Choosing between Gaussian and Trapezoidal Membership Functions in Interval Type-2 Fuzzy Logic Controllers

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***Abstract*—Interval type-2 fuzzy logic controllers (IT2 FLCs) have been attracting great research interests recently. There are many decisions to be made in designing an IT2 FLC. One of them is to determine which membership function type to use, e.g., Gaussian or trapezoidal. There have not been comprehensive studies on this problem so far. In this paper we present 12 considerations in choosing between Gaussian and trapezoidal membership functions for an IT2 FLC, including representation, construction, optimization, adaptiveness, novelty, analytical structure, continuity, monotonicity, stability, robustness, computational cost, and control performance. It can help practitioners select the appropriate membership function type in IT2 FLC design, and researchers identify new research opportunities on IT2 FLCs. Our study shows that each MF type has its own advantages: Gaussian IT2 FLCs are simpler in design because they are easier to represent and optimize, always continuous, and faster for small rulebases, whereas trapezoidal IT2 FLCs are simpler in analysis.**

***Index Terms*—Interval type-2 fuzzy logic controller, adaptiveness, novelty, analytical structure, continuity, monotonicity, stability, robustness, computational cost**

I. Introduction

Interval type-2 fuzzy sets (IT2 FSs) and systems have been attracting great research interests recently [37], [39], [46], [72]. They are particularly popular in modeling and control, where they have demonstrated better abilities to handle uncertainties than their T1 counterparts [6], [16], [20], [28], [32], [65], [66]. Fig. 1 shows the schematic diagram of an IT2 fuzzy logic system. It is similar to its type-1 (T1) counterpart, the major difference being that at least one of the FSs in the rulebase is an IT2 FS. Hence, the outputs of the inference engine are IT2 FSs, and a type-reducer [21], [37] is needed to convert them into a T1 FS before defuzzification can be carried out. Type-reduction is usually performed by the iterative Karnik-Mendel (KM) algorithms [21].

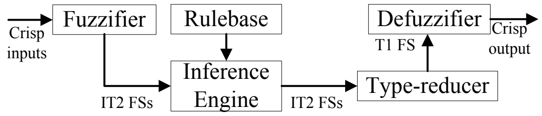

Fig. 1. The schematic diagram of an IT2 fuzzy logic system.

It is well-known [8] that T1 fuzzy logic controllers (FLCs) using a wide class of membership functions (MFs) are universal approximators, i.e., they are capable of approximating any real continuous function on a compact set to arbitrary accuracy. Ying [70], [71] has also shown that both Mamdani and TSK IT2 FLCs using any MF shapes are universal approximators. As pointed out by Castro [8], these results have two consequences: "*First, if we use a fixed type of FLC, the control is theoretically possible. Second, if we want to design a FLC, we should use the type which is more appropriate to that particular problem, because almost all types are theoretically effective.*" So, an important question is: which type of IT2 FLC is more appropriate to a particular problem?

There are many decisions to be made in designing an IT2 FLC, e.g., which MF shape to use, how many MFs in each input and output domain, how to construct the rulebase, how to reduce the computational cost, etc. In this paper we focus on the first problem, i.e., which MF shape should be used.

In the literature two MF shapes, Gaussian and trapezoidal (with triangular MFs as special cases), are most popular; so, only these two shapes are considered in this paper. Also, we focus on PI controllers because they are the most widely used controllers in practice. We consider its incremental form:

$$\dot{u} = k_P \dot{e} + k_I e \tag{1}$$

where $\dot{u}$ is the change in control signal, e is the error, $\dot{e}$ is the change of error, and k_P and k_I are proportional and integral gains, respectively. Additionally, we focus on the center-of-sets type-reduction and KM algorithms based type-reducers [37], because they are the most widely used approaches.

Twelve considerations in choosing between Gaussian and trapezoidal IT2 FSs are described next.

II. Twelve Considerations in Choosing between Gaussian and Trapezoidal MFs

This section presents 12 considerations in determining whether Gaussian or trapezoidal MFs should be used in an IT2 FLC, including representation, construction, optimization, adaptiveness, novelty, analytical structure, continuity, monotonicity, stability, robustness, computational cost, and control performance.

A. Representation

Representation concerns how to effectively and efficiently describe a MF. Generally a MF shape with simpler representations is preferred, especially if the parameters of the MF need to be optimized, because simpler representation usually means faster convergence.

A Gaussian T1 FS X is shown in Fig. 2(a) as the thick dashed curve. Its MF is:

$$\mu_X(x) = e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{2}$$

where $\mu_X(x)$ is the membership grade of x on X, m is the mean of the Gaussian FS, and σ is the standard deviation. Clearly, a T1 Gaussian FS has non-zero membership grades over the entire input domain. A Gaussian IT2 FS is usually obtained by blurring the mean or standard deviation of a baseline Gaussian T1 FS [65], as shown in Fig. 2. When the mean of the Gaussian T1 FS is blurred to be an interval $[m_1, m_2]$, as shown in Fig. 2(a), the UMF is

$$\mu_{\overline{X}}(x) = \begin{cases} e^{-\frac{(x-m_1)^2}{2\sigma^2}}, & x < m_1 \\ 1, & m_1 \le x \le m_2 \\ e^{-\frac{(x-m_2)^2}{2\sigma^2}}, & x > m_2 \end{cases} \tag{3}$$

and the LMF is

$$\mu_{\underline{X}}(x) = \min(e^{-\frac{(x-m_1)^2}{2\sigma^2}}, e^{-\frac{(x-m_2)^2}{2\sigma^2}}) \tag{4}$$

When the standard deviation of the Gaussian T1 FS is blurred to be an interval $[\sigma_1, \sigma_2]$, as shown in Fig. 2(b), the UMF is

$$\mu_{\overline{X}}(x) = e^{-\frac{(x-m)^2}{2\sigma_2^2}} \tag{5}$$

and the LMF is

$$\mu_{\underline{X}}(x) = e^{-\frac{(x-m)^2}{2\sigma_1^2}} \tag{6}$$

Observe that in either case, only three parameters $[(m_1, m_2, \sigma)$ or $(m, \sigma_1, \sigma_2)]$ are needed to define a Gaussian IT2 FS.

A trapezoidal T1 FS X is shown in Fig. 3 as the thick dashed curve. It is determined by four parameters (a', b', c', d'), and its MF is:

$$\mu_X(x) = \begin{cases} \frac{x-a'}{b'-a'}, & a' < x < b' \\ 1, & b' \le x \le c' \\ \frac{d'-x}{d'-c'}, & c' < x < d' \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

which may be written in a more concise form:

$$\mu_X(x) = \max\left(0, \min\left(\frac{x-a'}{b'-a'}, \frac{d'-x}{d'-c'}, 1\right)\right) \tag{8}$$

Note that triangular T1 FSs are special cases of trapezoidal T1 FSs when $b' = c'$.

A trapezoidal IT2 FS can also be obtained by blurring a baseline trapezoidal T1 FS [66], as shown in Fig. 3. Generally, nine parameters are needed to represent a trapezoidal IT2 FS, as $(a, b, c, d, e, f, g, i, h)$ shown in Fig. 3, where (a, b, c, d) determines the UMF and (e, f, g, i, h) determines the subnormal LMF.

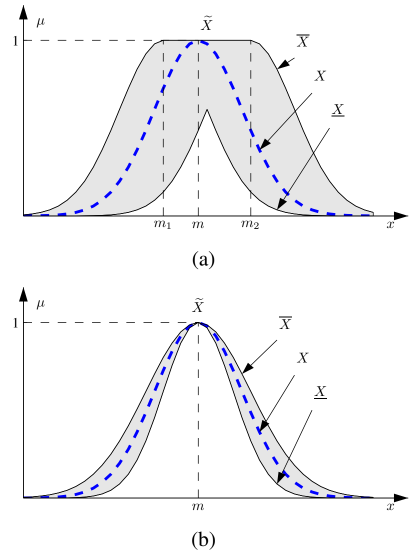

Fig. 2. Gaussian T1 and IT2 FSs. (a) A Gaussian T1 FS (thick dashed curve) and a Gaussian IT2 FS obtained from blurring the mean of the T1 FS; (b) a Gaussian T1 FS (thick dashed curve) and a Gaussian IT2 FS obtained from blurring the standard deviation of the T1 FS.

From the above description we can conclude that generally it is simpler to represent a Gaussian IT2 FS because it only needs three parameters, whereas a trapezoidal IT2 FS needs nine parameters.

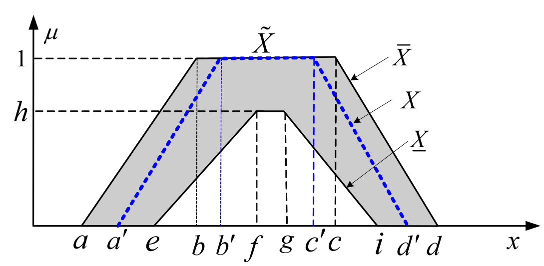

Fig. 3. A trapezoidal T1 FS (thick dashed curve) and a trapezoidal IT2 FS, represented by nine parameters.

B. Construction

Construction concerns the methods to obtain the MFs. A MF shape that can be constructed in more ways is more favorable.

Generally there are two approaches for constructing the FSs in a FLC: model-driven and knowledge-driven. For the model-driven approach, we have a mathematical model of the plant, so optimization algorithms [65], [66] can be used to tune the parameters of the MFs. Both Gaussian and trapezoidal MFs can be used in this approach. For the knowledge-driven approach [33], [34], the user specifies the rulebase and constructs the MFs according to his/her understanding of the linguistic terms. However, it is not easy to construct IT2 FS word models, especially if several people are working together and they have different opinions.

Recently Liu and Mendel [31] proposed an Interval Approach for constructing IT2 FSs from interval survey data, and Wu et al. [59] further improved upon it. In both approaches, for each word in an application-dependent encoding vocabulary, a group of subjects are asked the following question:

> *On a scale of 0-10, what are the end-points of an interval that you associate with the word ____?*

After some pre-processing, during which some intervals (e.g., bad data, outliers) are eliminated, each of the remaining intervals is classified as either an interior, left-shoulder or right-shoulder IT2 FS. Then, each of the word's data intervals is individually mapped into its respective T1 interior, left-shoulder or right-shoulder MF, after which the footprint of uncertainty of the IT2 FS is computed. So far only trapezoidal IT2 FSs can be generated from these approaches.

In summary, Gaussian IT2 FSs can only be constructed from the model-driven approach, whereas trapezoidal IT2 FSs can be constructed from both model-driven and knowledge-driven approaches. So, using trapezoidal IT2 FSs gives the user more freedom in MF construction.

C. Optimization

Optimization concerns the way to tune the parameters of the MFs. Generally a MF shape that can be optimized more efficiently is preferred.

In the literature there are two popular categories of methods to tune the parameters of IT2 FSs. The first consists of steepest descent algorithms (also referred to as back-propagation algorithms) [37], [40], [50], and the second consists of evolutionary computation algorithms [6], [50], [65], [66], especially genetic algorithms. The steepest descent algorithms need to compute the derivatives of the MF parameters. This is challenging for both Gaussian and trapezoidal IT2 FSs because the KM type-reducer does not have a closed-form solution. One example on IT2 FLCs using Gaussian MFs with uncertain means is given by Mendel [38]. Because of this, optimization of IT2 FLCs using evolutionary computation algorithms are more popular in the literature and practice. Gaussian IT2 FSs are generally more favorable in this case because fewer parameters are needed to represent them.

In summary, it is challenging to optimize the parameters of both Gaussian and trapezoidal IT2 FSs using steepest descent algorithms. When evolutionary computation algorithms are used, Gaussian MFs are more favorable since they have fewer parameters to tune.

D. Adaptiveness

We [53], [55] considered adaptiveness as one of the two fundamental differences between IT2 and T1 FLCs. Adaptiveness means that the embedded T1 FSs used to compute the bounds of the type-reduced interval change as input changes.

Consider an IT2 FLC [55] with two inputs (x_1 and x_2) and one output (y). Each input domain consists of two trapezoidal IT2 FSs, shown as the shaded areas in Fig. 4. The rulebase consists of the following four rules:

R^1 : IF x_1 is $\widetilde{X}_{11}$ and x_2 is $\widetilde{X}_{21}$, THEN y is Y^1.

R^2 : IF x_1 is $\widetilde{X}_{11}$ and x_2 is $\widetilde{X}_{22}$, THEN y is Y^2.

R^3 : IF x_1 is $\widetilde{X}_{12}$ and x_2 is $\widetilde{X}_{21}$, THEN y is Y^3.

R^4 : IF x_1 is $\widetilde{X}_{12}$ and x_2 is $\widetilde{X}_{22}$, THEN y is Y^4.

The corresponding rule consequents are given in Table I.

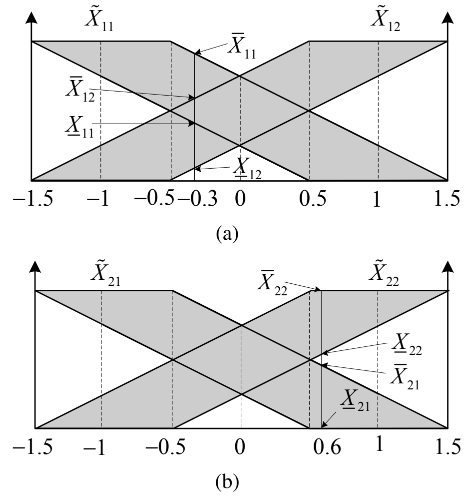

Fig. 4. MFs of the IT2 FLC. (a) Input MFs of x_1; (b) Input MFs of x_2.

TABLE I
RULEBASE AND CONSEQUENTS OF THE IT2 FLC.

x_1 \ x_2	$\widetilde{X}_{21}$	$\widetilde{X}_{22}$
$\widetilde{X}_{11}$	$Y^1 = [\underline{y}^1, \overline{y}^1] = [-1, -0.9]$	$Y^2 = [\underline{y}^2, \overline{y}^2] = [-0.6, -0.4]$
$\widetilde{X}_{12}$	$Y^3 = [\underline{y}^3, \overline{y}^3] = [0.4, 0.6]$	$Y^4 = [\underline{y}^4, \overline{y}^4] = [0.9, 1]$

Denote the firing interval of Rule R^i as $[\underline{f}^i, \overline{f}^i]$, and the type-reduced interval, computed by the KM algorithms, as $[y_l, y_r]$. When the input is $(x_1', x_2') = (-0.3, 0.6)$, according to the KM algorithms [55],

$$y_l = \frac{\overline{f}^1 \underline{y}^1 + \underline{f}^2 \underline{y}^2 + \underline{f}^3 \underline{y}^3 + \underline{f}^4 \underline{y}^4}{\overline{f}^1 + \underline{f}^2 + \underline{f}^3 + \underline{f}^4} \tag{9}$$

When the input is $(x_1', x_2') = (0.3, 0.6)$,

$$y_l = \frac{\overline{f}^1 \underline{y}^1 + \overline{f}^2 \underline{y}^2 + \underline{f}^3 \underline{y}^3 + \underline{f}^4 \underline{y}^4}{\overline{f}^1 + \overline{f}^2 + \underline{f}^3 + \underline{f}^4} \tag{10}$$

The corresponding embedded T1 FSs used in (9) and (10) are shown in Table II. Observe that when the input changes from $(-0.3, 0.6)$ to $(0.3, 0.6)$, different embedded T1 FSs are used in computing the four firing levels and hence y_l. This is the adaptiveness that does not exist in traditional T1 FLCs.

Though the adaptiveness of IT2 FLCs is illustrated by trapezoidal MFs above, it is fundamental to IT2 FLCs and is independent of the shape of the MFs. In other words, both Gaussian and trapezoidal IT2 FSs preserve the adaptiveness of IT2 FLCs.

TABLE II
THE EMBEDDED T1 FSS FROM WHICH THE FOUR FIRING LEVELS IN (9) AND (10) ARE OBTAINED.

		$\tilde{X}_{11}$		$\tilde{X}_{12}$		$\tilde{X}_{21}$		$\tilde{X}_{22}$	
		UMF	LMF	UMF	LMF	UMF	LMF	UMF	LMF
Equation (9) $(x_1', x_2') = (-0.3, 0.6)$	$\overline{f}^1$	√				√			
	$\underline{f}^2$		√						√
	$\underline{f}^3$				√		√		
	$\underline{f}^4$				√				√
Equation (10) $(x_1', x_2') = (0.3, 0.6)$	$\overline{f}^1$	√				√			
	$\overline{f}^2$	√						√	
	$\underline{f}^3$				√		√		
	$\underline{f}^4$				√				√

E. Novelty

We [53], [55] considered novelty as another fundamental differences between IT2 and T1 FLCs. Novelty means that the upper and lower MFs of the same IT2 fuzzy set may be used simultaneously in computing each bound of the type-reduced interval.

Take y_l in (9) as an example. The firing levels of the four rules are $\overline{f}^1$, $\underline{f}^2$, $\underline{f}^3$ and $\underline{f}^4$, respectively, which are computed from different lower and upper MFs, as shown in the first part of Table II. Observe that both the upper and lower MFs of $\tilde{X}_{11}$ are used in computing y_l, and they are used in different rules: The UMF of $\tilde{X}_{11}$ is used in computing $\overline{f}^1$, the firing level of Rule R^1, whereas the LMF of $\tilde{X}_{11}$ is used in computing $\underline{f}^2$, the firing level of Rule R^2. Similarly, the upper and lower MFs of $\tilde{X}_{21}$ are used simultaneously in different rules for computing y_l. Observe also from the second part of Table II that the upper and lower MFs of $\tilde{X}_{21}$ and $\tilde{X}_{22}$ are used simultaneously in different rules for computing y_l. This novelty is impossible for a traditional T1 FLC, where the same MFs are always used in computing the firing levels of all rules.

Though the novelty of IT2 FLCs is illustrated by trapezoidal MFs above, it is fundamental to IT2 FLCs and is independent of the shape of the MFs. In other words, both Gaussian and trapezoidal IT2 FSs preserve the novelty of IT2 FLCs.

F. Analytical Structure

To better understand the characteristics of an IT2 FLC, several researchers [11], [42], [44], [67], [74] have tried to derive the analytical structure of IT2 FLCs. All of them used trapezoidal IT2 FSs and showed that a trapezoidal IT2 PI FLC is equivalent to a variable-gain PI controller. The value and functional representations of the variable PI gains change as the inputs [$e(t)$ and/or $\dot{e}(t)$] change.

Zhou and Ying [75] presented a comprehensive study on deriving the analytical structure of a broad class of IT2 Mamdani FLCs, where the MFs can have arbitrary shapes. The equivalent variable-gain nonlinear controller can be expressed as:

$$\begin{aligned}\dot{u}(t) =& k_I'(e(t),\dot{e}(t)) \cdot e^2(t) + k_P(e(t),\dot{e}(t)) \cdot \dot{e}(t) \\ &+ k_I(e(t),\dot{e}(t)) \cdot e(t) + \delta(e(t),\dot{e}(t)) \end{aligned} \quad (11)$$

where $k_I'(e(t),\dot{e}(t))$, $k_P(e(t),\dot{e}(t))$, $k_I(e(t),\dot{e}(t))$ and $\delta(e(t),\dot{e}(t))$ are functions of $e(t)$ and $\dot{e}(t)$, and they may have different representations when $e(t)$ and $\dot{e}(t)$ change. They showed that if and only if all the input FSs are piecewise linear, e.g., when the input FSs are trapezoidal IT2 FSs, $k_I'(e(t),\dot{e}(t))$ in (11) becomes 0 and the IT2 PI FLC becomes a nonlinear PI controller with variable PI gains plus a variable offset term. For Gaussian IT2 FLCs, $k_I'(e(t),\dot{e}(t))$ in (11) is nonzero and hence it has a $e^2(t)$ term. As a result, it is not a PI controller, which makes its performance analysis more challenging.

In summary, trapezoidal IT2 FLCs have simpler analytical structures than Gaussian IT2 FLCs, though they are still very complex.

G. Continuity

Continuity is a very important and desirable property for FLCs because most physical systems are continuous, and a continuous and smooth control surface is usually more favorable in terms of stability and performance [20], [58], [62], [64], [65]. We [58] have done a comprehensive study on the continuity of T1 and IT2 FLCs. Two types of discontinuities were considered. A function $f(x)$ has a *gap discontinuity* at c if $f(c)$ is *undefined*. For example, $f_1(x)/f_2(x)$ has a gap discontinuity at c if $f_2(c) = 0$. A function $f(x)$ has a *jump discontinuity* at c if $f(c)$ is *defined* but $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$, i.e., both $f(c)$ and $f(c+\delta)$ are defined, but $f(c+\delta)$ does not approach $f(c)$ as δ approaches 0. For example, $f(x) = \begin{cases} 2, & x<0 \\ 3, & x\geq 0\end{cases}$ has a jump discontinuity at $x = 0$.

We [58] have studied the continuity of IT2 fuzzy logic systems with an arbitrary number of inputs. Here we present the results for IT2 PI FLCs with only two inputs.

Result 1: An IT2 PI FLC has *gap discontinuity* if and only if there exist at least one point in the input domain that is not covered by the UMFs.

Result 2: An IT2 PI FLC has *jump discontinuity* if 1) the input domain is fully covered by the UMFs; 2) there exists at least one point not covered by the LMFs; and, 3) All rules have different consequents.

Result 3: An IT2 PI FLC is *continuous* as long as its input domain is fully covered by *both* the UMFs and the LMFs.

Based on the above results, we have the following guidelines for practitioners who want to design continuous IT2 PI FLCs:

1) To guarantee a continuous control surface regardless of which type-reduction and defuzzification method is used, Gaussian IT2 FSs should be employed.
2) When trapezoidal IT2 FSs are used, to guarantee a continuous control surface, the LMFs should cover every input domain. This implies that the UMFs must also cover every input domain.

Note that the constraints on the trapezoidal IT2 FSs are very easy to satisfy, so they have little impact on the performance and design freedom of trapezoidal IT2 FLCs.

In summary, Gaussian IT2 PI FLCs are always continuous, whereas trapezoidal IT2 PI FLCs are only continuous under certain easy constraints.

H. Monotonicity

Many real-world systems are monotonic, e.g., the fan speed of an air conditioner increases monotonically as the electricity current increases. This requires that the corresponding controller should also be monotonic. As pointed out in [49], the monotonicity property represents additional qualitative information/knowledge that can be exploited to obtain interpretable and optimized FLCs.

The monotonicity of T1 FLCs using both Gaussian and trapezoidal MFs have been studied by several researchers [5], [47], [49], [52]. The goal is to derive the conditions under which a T1 FLC is guaranteed to be monotonic. There are only a few studies [26], [27], [51] on the monotonicity of IT2 FLCs because of its complexity. All of them considered only trapezoidal or triangular IT2 FSs. They also have other limitations. Li et al. [26] studied the monotonicity of an IT2 FLC using the KM type-reduction algorithms; however, only single-input single-output IT2 FLCs were considered. Wang et al. [51] studied the monotonicity of a single-input single-output IT2 FLC whose output is un-normalized, whereas most IT2 FLCs used in practice are normalized. Li et al. [27] also considered single input rule modules (SIRMs) connected IT2 FLCs[1], which can have multiple inputs; however, their type-reduction method is different from the most widely used KM algorithms.

In summary, there have been some preliminary studies on the monotonicity of IT2 FLCs using trapezoidal MFs, but no studies on the monotonicity of IT2 FLCs using Gaussian MFs. So far there are no results to guarantee that a Gaussian or trapezoidal IT2 FLC using the KM algorithms is monotonic.

I. Stability

Stability is very important for a controller because it is related to safety. There have been several studies on the stability of IT2 FLCs [2], [7], [18], [19], [23], [24], which include methods for designing stable IT2 FLCs and methods for testing whether an IT2 FLC is stable or not.

Castillo et al. [7] extended Margaliot and Langholz's [35] method for designing stable Mamdani T1 FLCs to IT2 FLCs. They demonstrated the concept using a very simple example, where only two trapezoidal IT2 FSs were used in each of the two input domains. However, there are some limitations with their approach. First, they required that the control signal is proportional to a state variable or its derivative, which is not always true in practice. Second, they translated the Lyapunov condition $x_2(x_1 + \dot{x}_2) < 0$ to rules like "If x_1 is positive and x_2 is positive, Then $\dot{x}_2$ must be negative big" so that each individual rule is stable. To ensure the aforementioned rule always satisfies the Lyapunov condition, $|\dot{x}_2|$ must be larger than the largest element of the support of the FS "positive" (otherwise $x_1 + \dot{x}_2$ still has a chance to be positive, and hence $x_2(x_1 + \dot{x}_2) > 0$). Clearly, this is not possible for Gaussian MFs. Though this is possible for trapezoidal MFs, it imposes a significant constraint on how the FSs can be defined. We believe the constraints can be relaxed if stability can be considered at the fused output level instead of individual rule level.

[1]A SIRMs based IT2 FLC constructs a single-antecedent IT2 FLC for each input, computes the outputs of such single-antecedent IT2 FLCs separately using the KM algorithms, and then combines the outputs using a weighted average.

Lam and Seneviratne [23] studied the stability of IT2 FLCs using linear matrix inequalities and the Lyapunov stability theory. One nice part of their approach is that the MFs can be any forms including Gaussian and trapezoidal. However, the plant under control must also have a fuzzy logic model, and the IT2 FLC must have the same antecedent MFs and the same number of rules as the plant. In their more recent work [22] these constraints were removed, and hence design flexibility was enhanced. However, they indicated that this leads to conservative stability analysis results. It also needs to point out that in both studies they used a normalized central FLC whose type-reduction and defuzzification method is different from the popular iterative KM algorithms.

Biglarbegian et al. [2] studied the sufficient conditions for the stability of IT2 TSK FLCs whose MFs can have arbitrary shapes including Gaussian and trapezoidal. They also used linear matrix inequalities and the Lyapunov stability theory. To make the analysis more feasible, they employed a closed-form type-reduction and defuzzification method, which again is different from the popular iterative KM algorithms.

Li et al. [24] performed stability analysis for the SIRMs based IT2 FLCs, in which both Gaussian and trapezoidal IT2 FSs can be used; however, its structure is different from traditional IT2 PI FLCs. Additionally, they only studied local stability around the equilibrium point because it is very difficult to study the global stability.

Jafarzadeh et al. [18], [19] proposed sufficient conditions for the exponential stability of T1 and general type-2 TSK FLCs. A major advantage of their approach is that it doe not require the existence of a common Lyapunov function and is therefore applicable to systems with unstable consequents. However, their approach only applies to special triangular T1 FSs which are orthogonal, consistent, complete and normal, and special triangular IT2 FSs whose UMFs are orthogonal, consistent, complete and normal and whose LMFs are scaled and shifted versions of the UMFs.

In summary, there have been some studies on the stability of both Gaussian and trapezoidal IT2 FLCs; however, each approach has its own limitations, and so far there has not been comprehensive stability analysis for either Gaussian or trapezoidal IT2 FLCs using the popular KM type-reduction algorithms. So, it is difficult to conclude which MF shape is more favorable in terms of stability.

J. Robustness

Robustness is another important property of controllers. As pointed out by Biglarbegian et al. [4], "*when a system* *is subjected to small deviations around the sampling points (operating points), it is essential to find the maximum tolerance of the system with respect to those perturbations, referred to herein as the systems robustness. Thus, in the context of modeling, robustness is a metric for measuring the impact of input deviations on the desired output.*"

Many experiments have verified that IT2 FLCs are more robust than their T1 counterparts [16], [30], [65], [66]. However, only a few researchers [3], [4], [24] have tried to investigate their robustness directly. Biglarbegian et al. [3], [4] studied the robustness of IT2 FLCs using an alternative type-reduction method, which has closed form solution and is different from the iterative KM algorithms. Their method can be applied to both Gaussian and trapezoidal IT2 FSs. Li et al. [24] studied the robustness of the SIRMs connected IT2 FLCs and illustrated their method using triangular IT2 FSs. Their method can also be applied to trapezoidal or Gaussian IT2 FSs. However, they also used an alternative type-reduction method which is an extension of Biglarbegian et al.'s method. The robustness of IT2 FLCs using the most popular KM type-reduction algorithms remains unexplored.

In summary, there have been some studies on the robustness of IT2 FLCs using alternative closed-form type-reducers, but no studies on IT2 FLCs using the KM type-reducer. Hence, it is difficult to conclude which MF shape will result in more robust IT2 FLCs when the KM type-reducer is used.

K. Computational Cost

The iterative KM type-reduction algorithms [21] are computationally intensive and may hinder IT2 FLCs from certain real-time applications. There have been many different approaches for reducing the computational cost of IT2 FLCs. We [54] presented a comprehensive overview and comparison of them. These approaches can be grouped into three categories:

1) *Enhancements to the KM type-reduction algorithms* [12], [17], [36], [57], [60], [69], which improve directly over the original KM algorithms to speed them up. We [54] gave an overview and comparison of five such enhancements and showed that for practice PI FLCs (i.e., the rulebase has less than 100 rules) the enhanced opposite direction searching algorithms [17] are the fastest.
2) *Alternative type-reduction algorithms* [1], [10], [11], [13], [14], [25], [29], [43], [48], [63], [68]. Unlike the iterative KM algorithms, these alternative type-reduction algorithms have closed-form representations. They are usually fast approximations of the KM algorithms. We [56] gave an overview and comparison of 11 such approaches. Experiments demonstrated that ten of them are generally faster than the KM algorithms; among them, the Wu-Tan [63] and Nie-Tan [43] methods are the fastest, and they are only about 1.2-1.7 times slower than a T1 FLC.
3) *Simplified IT2 FLCs* [61], [66], in which the architecture of an IT2 FLC is simplified by using only a small number of (usually only one) IT2 FSs for the most critical input regions and T1 FSs for the rest. In [61], a simplified IT2 FLC, which used one IT2 FS around $e = 0$ and one around $\dot{e} = 0$, outperformed a T1 FLC with the same number of MFs and showed similar performance as an IT2 FLC whose MFs are all IT2 FSs. In [66], a simplified IT2 FLC, which used only one IT2 FS in the $\dot{e}$ domain, outperformed a T1 FLC with the same number of MFs and showed similar performance as a T1 FLC with more MFs and an IT2 FLC whose all MFs are IT2 FSs.

All methods in these three categories can be applied to both Gaussian and trapezoidal IT2 FLCs. Our experimental results [54] showed that when the rulebase is small (e.g., there are less than 50 rules), generally Gaussian IT2 FLCs are faster than the corresponding trapezoidal IT2 FLCs when the same number of MFs and the same type-reduction and defuzzification method are used. Since small rulebases are usually used in practice, Gaussian IT2 FLCs seem more favorable in terms of computational cost.

L. Control Performance

Control performance is the most important consideration in choosing between Gaussian and trapezoidal MFs in a FLC. There are lots of studies on comparing the control performance of Gaussian and trapezoidal MFs in T1 FLCs [9], [15], [41], [45], [73]; however, it seems that the conclusion is highly application dependent, and it is difficult to conclude which MF shape is always better.

To the best knowledge of the author, so far there has not been any study on directly comparing the performance of Gaussian and trapezoidal MFs in IT2 FLCs. Wu [65] evaluated the performance of IT2 FLCs using Gaussian MFs on a coupled-tank liquid level control system, and also the performance of IT2 FLCs using trapezoidal MFs on the same system in a different publication [66]. Both evaluations showed that IT2 FLCs can outperform their type-1 counterparts; however, no direct comparisons between the performance of IT2 FLCs using different MF shapes were made. We conjecture that like the T1 FLC case, which IT2 FS shape is better is also application dependent, and it will not be known until the IT2 FLCs using different MF shapes are designed and tested.

In summary, it is difficult to predict whether a Gaussian or a trapezoidal IT2 FLC will give better control performance for a particular application: we need to design and compare them. However, if the budget permits only one IT2 FLC to be designed, then the eleven considerations presented above will be very important in determining which IT2 FLC to start with.

M. Summary

For the convenience of the readers, the 12 considerations presented in this section are summarized in Table III. Clearly, there are lots of research opportunities on IT2 FLCs, especially, their monotonicity, stability, robustness, and direct comparative studies on the effects of different MF shapes. So, it is difficult to conclude which MF type is better. Each MF type has its own advantages: Gaussian IT2 FLCs are simpler in design because they are easier to represent and optimize, always continuous, and faster for small rulebases, whereas trapezoidal IT2 FLCs are simpler in analysis.

III. Conclusions

IT2 FLCs have been attracting great research interests recently. One of the most important decisions to be made in designing an IT2 FLC is to determine which MF type to use, e.g., Gaussian or trapezoidal. In this paper we have presented 12 considerations in choosing between Gaussian and trapezoidal MFs, including representation, construction, optimization, adaptiveness, novelty, analytical structure, continuity, monotonicity, stability, robustness, computational cost, and control performance. Each MF type has its own advantages: Gaussian IT2 FLCs are simpler in design because they are easier to represent and optimize, always continuous, and faster for small rulebases, whereas trapezoidal IT2 FLCs are simpler in analysis. This paper will be very useful to practitioners on IT2 FLC design, and to researchers who want to identify new research opportunities on IT2 FLCs.

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TABLE III
A SUMMARY OF THE 12 CONSIDERATIONS IN CHOOSING BETWEEN GAUSSIAN AND TRAPEZOIDAL IT2 FSS.

Consideration	Gaussian IT2 FS	Trapezoidal IT2 FS
Representation	Three parameters for each IT2 FS	Nine parameters for each IT2 FS
Construction	Only model-driven approach can be used	Both model-driven and knowledge-driven approaches can be used
Optimization	Fewer parameters to tune	More parameters to tune
Adaptiveness	Preserved	Preserved
Novelty	Preserved	Preserved
Analytical Structure	More complex than a PI controller	Equivalent to a variable gain PI controller plus a variable offset
Continuity	Always continuous	Continuous under certain easy constraints
Monotonicity	No studies yet	Preliminary studies
Stability	No studies on IT2 FLCs using the KM type-reducer; difficult to conclude which is more stable	
Robustness	No studies on IT2 FLCs using the KM type-reducer; difficult to conclude which is more robust	
Computational Cost	Many approaches for reducing the computational cost; generally Gaussian IT2 FLCs are faster for small rulebases	
Control Performance	No direct comparative studies so far; may be application-dependent	

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