Information Sciences 517 (2020) 37–51

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Set-Membership filtering with incomplete observations

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ARTICLE INFO

Article history:
Received 3 December 2018
Revised 8 December 2019
Accepted 30 December 2019
Available online 31 December 2019

Keywords:
Set-membership estimation
Stability analysis
Incomplete observations
Discrete time-varying systems

ABSTRACT

This study addresses the set-membership estimation problem for a class of discrete time-varying systems with incomplete observations. A set-membership filter is developed and a recursive algorithm is proposed to calculate the state estimate ellipsoid which contains the true value. To solve the problem that the conventional set-membership filter cannot guarantee the stability when applied to discrete time-varying systems with incomplete observations, a quantitative analysis method about incomplete observations is developed and a tight bound of the estimation error is found based on interval analysis and some bounded noise assumptions. In terms of bounded assumptions, the relationship between the bound of estimated error and the data dropout rate is obtained. If the data dropout rate is less than a maximal value, the set-membership filter is asymptotically stable. The proposed filter is applied to a two-state example to demonstrate the effectiveness of theoretical results.

1. Introduction

It is well known that the commonly used Kalman filtering algorithm requires the process and noise to be white Gaussian processes [17]. This noise model is widely applied in a lot of applications and has obtained great success. Sometimes we only know that the process and measurement noises are unknown but bounded (UBB) (e.g. [4,31]). To follow the set based noise model, it is natural to consider all the possible state vectors into a set of state estimates in which the true states are contained. In this case, the result of estimation is a set rather than a vector. This kind of estimation problem is generalized as a set-membership filtering problem (e.g. [3,4,16,31,37–39]).

It should be noted that the possible state estimation (such as the set-membership filtering) is a kind of recursive state-bounding approach. After receiving the successive observations, the feasible sets which include all the possible state values consistent with the state and observation equations are updated. For linear systems, if the exact additive state noise bounds are known, we can calculate an exact convex polytope in the state space. Unfortunately, its computational complexity grows rapidly with observations updates (e.g. [5]). Various methods can be applied to reduce the computational complexity. For instance, to limit the feasible set in simple polytopes such as parallelotopes (e.g. [7]). Another way to avoid increasing complexity is to apply the ellipsoidal approximation algorithm introduced in [31]. In this algorithm, the feasible set can be approximated by ellipsoids. Compared with other algorithms, the outer bounded ellipsoids based algorithm has the advantage of simplicity and computational efficiency (e.g. [12,18,20]).

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https://doi.org/10.1016/j.ins.2019.12.087

Recently, many researchers are interested in dealing with the set-membership filtering problems for various uncertain systems. A nonlinear set-membership filtering approach which applies the ellipsoidal approximation algorithm is developed in [30]. A computationally efficient algorithm for set-membership filtering with ellipsoidal approximations is introduced in [21]. A set-membership based framework for the designing of nonlinear system is proposed in [1]. A set-based estimation method for T-S fuzzy systems with unknown output delay signals is introduced in [43]. A set-membership identification approach is used in [15] to design a fault detection system for mating a pair of electric connectors. In [27], a set-membership identification method is used to control the power kites for wind energy conversion. Except for ellipsoids, there are also set-membership state estimation methods based other geometrical forms such as zonotopes [8], positive invariant sets [22], polytopes [34], etc.

It is assumed that the observation is always consecutive in much literature (see for example [19,40,41]). In applications, however, the observation cannot always be perfectly obtained. For example, packet loss in the networked control systems [29], or sensor failure in a real system, will both lead to the missing observations. The filtering problem with missing observations was first studied in [14] and [24]. Such a problem has attracted much attention during past few years because the network is widely used in modern applications. One of the modeling methods for missing data is using a binary switching sequence specified by a conditional probability distribution (see [14], [24] and [32]). Based on this model, the statistical convergence properties of Kalman filter with missing observations have been discussed in [32], and robust finite-horizon filtering with missing observations is presented in [35]. To the best of our knowledge, limited literature can be found on the topic of set-membership filtering with missing observations. Wei studied the probability-guaranteed set-membership filtering problem for a class of time-varying systems with incomplete measurements [37]. Yang addressed the robust set-membership filtering algorithm with missing observations [42]. In [9], the set based method with limited communication was proposed to design a self-triggered and event-triggered observer. At the same time, we noticed that the convergence of the algorithm was rarely discussed. There is still much room for improvement of the studies in this field. In [8], a zonotopic Kalman filter was proposed, and a bridge between the zonotopic set-membership and the stochastic paradigms for Kalman filtering was built. In [23], an H-infinity set-membership observer was proposed and the estimation error was bounded by an ellipsoid robustly positive invariant set. The advantage is that this approach does not need online computation of sets.

In this paper, we model the missing observations by an unknown but limited sequence. The realistic background of this missing observation model is motivated by the features of packet loss in intermittent communication. We usually do not know when the packet loss will happen. However, when the packet loss happens, it is reasonable to assume that the communication will be recovered within finite time. In other words, the maximum packet loss period is supposed to be known in this model. This work follows the line of optimal bounding ellipsoid (OBE) algorithm (e.g. [6,25]), which calculates the updated sets online and solves an optimization problem to guarantee that the volume is minimized. The advantage is that this class of algorithms has a relatively small estimated set compared with the original ellipsoid estimation approach in [31]. The main contributions of our work are: 1) the convergence of the proposed set-membership filter with incomplete observations is theoretically studied; 2) the effect of the missing observations to the estimation error of filtering algorithm is analyzed and some quantitative results are obtained.

The remainder of this paper is presented as follows. Several definitions used in this paper are described in Section 2. A set-membership filtering algorithm with incomplete observations is proposed in Section 3. The stability analysis of the proposed algorithm is given in Section 4. Section 5 firstly provides a numerical example to demonstrate the effectiveness of our algorithm, then compares our proposed algorithm to a related Kalman filter. Conclusions and future work are described in Section 6.

2. Notation

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{N}$ the set of natural numbers. For vectors in $\mathbb{R}^n$, we use $\|\cdot\|$ as the Euclidean norm.

Definition 2.1. The description of an ellipsoid is given by a set $\Omega \in \mathbb{R}^n$

$$\Omega = \{x : (x-a)^T P^{-1} (x-a) \le \sigma^2\},$$

where $a \in \mathbb{R}^n$ is the center of the ellipsoid, $x \in \mathbb{R}^n$ is an arbitrary possible value within the ellipsoid, and $P \in \mathbb{R}^{n\times n}$ is a positive definite matrix which determines the shape of the ellipsoid. $\sigma \in \mathbb{R}$ is not a physically interpretable measure of the size of the ellipsoid. It has been noted in [11] that σ is usually considered as a measure of optimality in OBE algorithms. We represent the ellipsoid as $\Omega(a, P, \sigma)$.

Definition 2.2. The vector sum ψ of two ellipsoids Ω_1 and Ω_2 is defined as

$$\psi = \{x : x = x_1 + x_2, x_1 \in \Omega_1, x_2 \in \Omega_2\}.$$

ψ is also described by $\psi = \Omega_1 \oplus \Omega_2$.

Definition 2.3. The outer bounding ellipsoid Ω_s is the ellipsoid which includes the vector sum ψ, i.e. we have $\psi \subseteq \Omega_s$. In the two-dimensional case, we can depict it as Fig. 1.

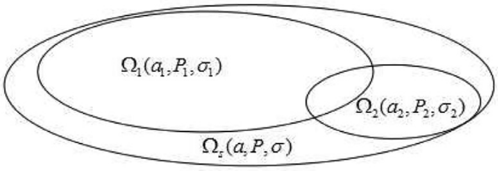

Fig. 1. The outer bounding ellipsoid of the vector sum in the two-dimensional case.

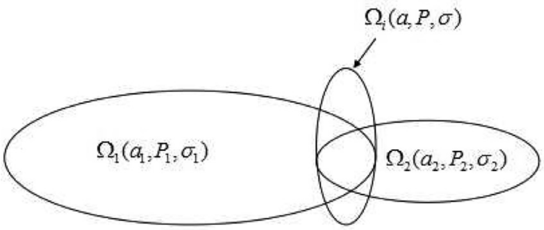

Fig. 2. The outer bounding ellipsoid of the intersection in the two-dimensional case.

Definition 2.4. The intersection Γ of two ellipsoids Ω_1 and Ω_2 is defined as $\Gamma = \{x : x \in \Omega_1 \cap x \in \Omega_2\}$. In what follows, Γ is denoted by $\Gamma = \Omega_1 \cap \Omega_2$.

Definition 2.5. The outer bounding ellipsoid Ω_i is the ellipsoid which includes the intersection set Γ, i.e. we have $\Gamma \subseteq \Omega_i$. In the two-dimensional case, we can depict it as Fig. 2.

Definition 2.6. The lower and upper bounds of time varying matrix X_k are defined as $\underline{x} \le ||X_k|| \le \bar{x}$, where the matrix norm $||X|| = \sqrt{\lambda_1}$, λ_1 is the largest eigenvalue of $X^T X$.

At time k, the estimated and predicted value of vector x_k are denoted by $\hat{x}_k$ and $x_{k|k-1}$, respectively.

Definition 2.7. The worst-case estimation error is defined as

$$||\hat{\Omega}_k||_{\max} \equiv \sup_{x \in \hat{\Omega}_k} ||x - \hat{x}_k|| = \hat{\sigma}_k \sqrt{\lambda_{\max}(P_k)},$$

where $\lambda_{\max}(P_k)$ is the maximum eigenvalue of matrix P_k.

3. Set-membership filter with incomplete observations

3.1. Problem setup

Consider the following class of linear discrete time-varying systems

$$x_{k+1} = A_k x_k + w_k, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector. $A_k \in \mathbb{R}^{n_x \times n_x}$ is a known time-varying matrix with appropriate dimensions and $w_k \in \mathbb{R}^{n_x}$ is the UBB noise which belongs to the given set $W_k = \Omega(0, Q_k, \bar{\sigma}_w)$, where Q_k is the known positive matrix and $\bar{\sigma}_w$ is a known positive scalar which represents the upper bound of the noise.

The output measurement, which may have missing data, is represented as follows

$$y_k = \Upsilon_k(C_k x_k + \upsilon_k), \tag{2}$$

where $y_k \in \mathbb{R}^{n_y}$ is the output vector of measures and usually $n_y < n_x$. $\{\Upsilon_k\}$ is an unknown binary sequence composed of 0 and 1. That is, y_k is available if $\Upsilon_k = 1$ and y_k is missing if $\Upsilon_k = 0$. $C_k \in \mathbb{R}^{n_y \times n_x}$ is also a known time-varying matrix with appropriate dimensions. $\upsilon_k \in \mathbb{R}^{n_y}$ is the sensor noise which belongs to the given set $\Omega(0, I, \gamma)$, where γ is a known positive scalar which represents the upper bound of the noise.

The initial state, x_0, is bounded by the ellipsoid $\Omega(\hat{x}_0, P_0, \sigma_0)$.

3.2. Time update

At time k, assume that the state vector x_k is bounded by an ellipsoid $\Omega(\hat{x}_k, P_k, \sigma_k)$.

According to (1), the state vector at time $k+1$ satisfies

$$x_{k+1} = \{x + w : x \in A_k\Omega_k, w \in W_k\}$$

where $A_k\Omega_k = \{x : (x_k - \widehat{x}_k)^T \widehat{P}_k^{-1} (x_k - \widehat{x}_k) \leq \widehat{\sigma}^2_k\}$, and $\widehat{x}_k = A_k\hat{x}_k$, $\widehat{P}_k = A_kP_kA_k^T$, $\widehat{\sigma}_k = \sigma_k$.

Based on the Appendix A of reference [20], considering ellipsoids Ω_1, Ω_2 and the outer bounding ellipsoid Ω which includes their vector sum, the following equation is satisfied

$$\Omega(a_1, Q_1, \sigma_1) \oplus \Omega(a_2, Q_2, \sigma_2) = \Omega(a_1 + a_2, Q(p), \sigma), \tag{3}$$

where $Q(p) = (1+p)\frac{\sigma_1^2}{\sigma^2}Q_1 + (1+p^{-1})\frac{\sigma_2^2}{\sigma^2}Q_2,\ \ p > 0$. The selection of parameter p determines the property of the outer bounding ellipsoid Ω. Parameter $\sigma > 0$ and can be selected arbitrarily.

The prediction ellipsoid containing the state at time $k+1$ is defined as $\Omega_{k+1|k} = \Omega(x_{k+1|k}, P_{k+1|k}, \sigma_{k+1|k})$.

From (3), it follows that

$$x_{k+1|k} = A_k\hat{x}_k, \tag{4}$$

$$P_{k+1|k} = (1+p_k)\frac{\sigma_k^2}{\sigma_{k+1|k}^2}A_kP_kA_k^T + (1+p_k^{-1})\frac{\bar{\sigma}_w^2}{\sigma_{k+1|k}^2}Q_k. \tag{5}$$

Since parameter $\sigma > 0$ and it can be selected arbitrarily, we select $\sigma_{k+1|k}^2 = \sigma_k^2$. Then we have

$$P_{k+1|k} = (1+p_k)A_kP_kA_k^T + (1+p_k^{-1})\frac{\bar{\sigma}_w^2}{\sigma_k^2}Q_k. \tag{6}$$

Remark 3.1. Actually, the selection of $\sigma_{k+1|k}^2$ has no influence on the shape of the prediction ellipsoid $\Omega_{k+1|k}$. The reason is illustrated in the following.

From the definition of $\Omega_{k+1|k}$ and (5), we have

$$\Omega_{k+1|k} = \{x : (x - x_{k+1|k})^T\left((1+p_k)\frac{\sigma_k^2}{\sigma_{k+1|k}^2}A_kP_kA_k^T + (1+p_k^{-1})\frac{\bar{\sigma}_w^2}{\sigma_{k+1|k}^2}Q_k\right)^{-1}(x - x_{k+1|k}) \leq \sigma^2{}_{k+1|k}\}.$$

After simple manipulation, it follows that

$$\Omega_{k+1|k} = \{x : (x - x_{k+1|k})^T((1+p_k)\sigma_k^2A_kP_kA_k^T + (1+p_k^{-1})\bar{\sigma}_w^2Q_k)^{-1}(x - x_{k+1|k}) \leq 1\}. \tag{7}$$

From (7) we know that the selection of $\sigma_{k+1|k}^2$ does not have any influence in the property of the prediction ellipsoid $\Omega_{k+1|k}$.

Remark 3.2. The selection of parameter p_k determines the property of ellipsoid $\Omega_{k+1|k}$. Traditionally, a minimum trace algorithm is provided in reference [6]. According to the method in [6], if the parameter p_k satisfies $p_k = \frac{\bar{\sigma}_w}{\sigma_k}\sqrt{\frac{tr(Q_k)}{tr(A_kP_kA_k^T)}}$, then the trace of matrix $P_{k+1|k}$ (which is a measure of the ellipsoid's volume) achieves its minimum. In order to guarantee the stability of the proposed set-membership filtering algorithm, we need to select a lower bound of the parameter p_k, which is discussed in Theorem 4.1.

3.3. Observation update

Due to the possible loss of measurements, we need to categorize the observation into two cases, i.e. the case where the observation is available and the case where the observation is missing.

3.3.1. Observation update when the observation is available at time $k+1$

In this case, $\Upsilon_{k+1} = 1$ holds. From (2) and the property of measurement noise, we know that the state vector is bounded by the following set with observation

$$\widehat{E}_{k+1} = \{x : ||y_{k+1} - C_{k+1}x||^2 \leq \gamma^2\}.$$

According to Proposition 2 in [26] and Theorem 2.1 in [10], the estimated ellipsoid Ω_{k+1} at time $k+1$ can be obtained from the linear combination of $\Omega_{k+1|k}$ and $\widehat{E}_{k+1}$. That is, we have

$$\begin{aligned}\Omega_{k+1} &= \{\Omega_{k+1|k} \cap \widehat{E}_{k+1}\} = \{x : (x_{k+1} - \hat{x}_{k+1})^TP_{k+1}^{-1}(x_{k+1} - \hat{x}_{k+1}) \leq \sigma^2{}_{k+1}\} \\ &= \{x : (x - x_{k+1|k})^TP_{k+1|k}^{-1}(x - x_{k+1|k}) + q_{k+1}||y_{k+1} - C_{k+1}x||^2 \leq \sigma^2{}_{k+1|k} + q_{k+1}\gamma^2\},\end{aligned}$$

where q_{k+1} is the parameter which determines the volume of ellipsoid Ω_{k+1} satisfying $q_{k+1} \geq 0$. From the mathematical derivation proposed in [25], we have:

$$P_{k+1}^{-1} = P_{k+1|k}^{-1} + q_{k+1}C_{k+1}^TC_{k+1}, \tag{8}$$

$$\sigma_{k+1}^2 = \sigma_{k+1|k}^2 + q_{k+1}\gamma^2 - q_{k+1}e_{k+1}^T S_{k+1}^{-1} e_{k+1}, \tag{9}$$

$$\hat{x}_{k+1} = x_{k+1|k} + q_{k+1}P_{k+1}C_{k+1}^T e_{k+1}, \tag{10}$$

where $S_{k+1} = I + q_{k+1}C_{k+1}P_{k+1|k}C_{k+1}^T$, and $e_{k+1} = y_{k+1} - C_{k+1}x_{k+1|k}$.

Remark 3.3. The optimal problem of selecting parameter q_{k+1} also has been discussed in [26], the objective of optimization is to minimize σ_{k+1}^2 with respect to parameter q_{k+1}. According to the optimal bounding ellipsoid(OBE) algorithm which is introduced in [26], if $q_{k+1} = q_{k+1}^*$ satisfying

$$q_{k+1}^* = \begin{cases} 0 & , e_{k+1} \le \gamma \\ \frac{1}{g_{k+1}}(\frac{|e_{k+1}|}{\gamma} - 1), & e_{k+1} > \gamma, \end{cases} \tag{11}$$

where g_{k+1} is the maximum singular value of $C_{k+1}P_{k+1|k}C_{k+1}^T$, then σ_{k+1}^2 achieves its minimum.

3.3.2. Observation update when the observation is missing at time $k+1$

In this case we directly choose the ellipsoid Ω_{k+1} as $\Omega_{k+1|k}$. That is, we have $P_{k+1} = P_{k+1|k}$, $\sigma_{k+1}^2 = \sigma_{k+1|k}^2$ and $\hat{x}_{k+1} = x_{k+1|k}$.

3.3.3. Set-membership filtering algorithm with incomplete observations

Now we summarize our set-membership filter with incomplete observation as algorithm 1.

Algorithm 1 Set-membership filtering algorithm with incomplete observations.

Require: $\hat{x}_k$, A_k, σ_k, $\bar{\sigma}_w$, P_k, Q_k, Υ_{k+1}, γ
1: Calculate the center of prediction ellipsoid $x_{k+1|k}$ from (4)
2: Select the parameter $p_k = \frac{\bar{\sigma}_w}{\sigma_k}\sqrt{\frac{tr(Q_k)}{tr(A_k P_k A_k^T)}}$
3: Let $\sigma_{k+1|k} = \sigma_k$
4: Calculate $P_{k+1|k}$ from (6)
5: **if** $\Upsilon_{k+1} = 1$ **then**
6: Select the parameter q_k from (11)
7: Calculate P_{k+1}, σ_{k+1}, $\hat{x}_{k+1}$ from (8), (9), (10)
8: **else**
9: **if** $\Upsilon_{k+1} = 0$ **then**
10: Calculate $P_{k+1} = P_{k+1|k}$, $\sigma_{k+1}^2 = \sigma_{k+1|k}^2$ and $\hat{x}_{k+1} = x_{k+1|k}$
11: **end if**
12: **end if**
13: **return** $\hat{x}_{k+1}, \sigma_{k+1}, P_{k+1}$.

4. Main results

4.1. Stability analysis if observations are available all the time

First of all, we rewrite (6), (8) and (10) as follows:

$$P_{k+1|k} = \alpha_k A_k P_k A_k^T + Q_k^*, \tag{12}$$

$$P_{k+1} = (I - K_{k+1}C_{k+1})P_{k+1|k}, \tag{13}$$

$$\hat{x}_{k+1} = x_{k+1|k} + K_{k+1}(y_{k+1} - C_{k+1}x_{k+1|k}), \tag{14}$$

where

$$K_{k+1} = q_{k+1}P_{k+1}C_{k+1}^T = P_{k+1|k}C_{k+1}^T\left(\frac{1}{q_{k+1}}I + C_{k+1}P_{k+1|k}C_{k+1}^T\right)^{-1},$$

and $\alpha_k = 1 + p_k$, $Q_k^* = (1 + p_k^{-1})\frac{\bar{\sigma}_w^2}{\sigma_k^2}Q_k$.

In order to prove the asymptotical stability of the proposed set-membership filter when there is no noise ($w_k = v_k = 0$), we introduce a result in reference [28]. Considering two non-singular square matrices M and N and $(M + N^{-1})$ is invertible, the following equation holds

$$(M^{-1} + N)^{-1} = M - M(M + N^{-1})^{-1}M. \tag{15}$$

The following lemma gives the relationship between $P_{k+1|k}$ and $P_{k|k-1}$, which is necessary for the analysis in the following theorem.

Lemma 4.1. *If the discrete time-varying system (1) and (2) is observable, (A_k, C_k) is in the form of Luenberger second observable canonical form, then $A_k(I - K_kC_k)$ is invertible and*

$$P^{-1}_{k+1|k} \le \alpha_k^{-1} A_k^{-T} (I - K_kC_k)^{-T} \cdot [P^{-1}_{k|k-1} - P^{-1}_{k|k-1}(P^{-1}_{k|k-1} + \alpha_k (I - K_kC_k)^T A_k^T Q_k^{*-1} A_k (I - K_kC_k))^{-1} P^{-1}_{k|k-1}] \cdot (I - K_kC_k)^{-1} A_k^{-1}. \tag{16}$$

Sketch of the Proof: This inequality comes from equalities (13). At first, we must show that $A_k(I - K_kC_k)$ is invertible. Assuming that A_k and C_k follow the Luenberger second observable canonical form, it is known that $A_k(I - K_kC_k)$ follows a full rank matrix form. By applying (15) and some transformation, we can obtain the inequality (16). Detailed proof could be found in the Appendix 6.1.

Next some assumptions used in this study are given.

Assumption 4.1. The UBB noise is bounded by $||w_k|| \le \bar{w}$, $\underline{q} \le ||Q_k^*|| \le \bar{q}$, $||v_k|| \le \bar{v}$. The system matrix is bounded by $\underline{a} \le ||A_k|| \le \bar{a}$, $\underline{c} \le ||C_k|| \le \bar{c}$.

Assumption 4.2. The discrete time-varying systems (1) and (2) are observable.

Based on the two assumptions and Lemma 2 in reference [30], if P_0 is a positive define matrix, then there exist real numbers $\bar{s}$ and $\underline{s}$ such that $\underline{s} \cdot I \le P_k \le \bar{s} \cdot I$ and $\underline{s} \cdot I \le P_{k|k-1} \le \bar{s} \cdot I$. We know that $K_k = q_k P_k C_k^T$, and then $I - K_kC_k$ is also bounded. We write that $\underline{k}^* \le ||I - K_kC_k|| \le \bar{k}^*$. The state estimation error is written as $\zeta_k = x_k - x_{k|k-1}$. Now we are ready to present the first main result.

Theorem 4.1. *Suppose Assumptions 4.1 and 4.2 are satisfied, (A_k, C_k) is in the Luenberger second observable canonical form. Then the estimation error of the set-membership filtering algorithm 1 will converge to a bound ε_1 satisfying*

$$\varepsilon_1 = \frac{\kappa_3 + \sqrt{\kappa_3^2 + 4\kappa_1\kappa_2}}{2\kappa_2},$$

where $\kappa_1 = \frac{\bar{n}^2}{\underline{s}}$, $\kappa_2 = \frac{1-\alpha_k^{-1}}{\bar{s}} + \frac{1}{\alpha_k \bar{s}^2 (\underline{s}^{-1} + \alpha_k \bar{k}^{*2} \bar{a}^2/\underline{q})}$, $\kappa_3 = \frac{2\bar{a}\bar{n}\bar{k}^*}{\underline{s}}$. *$\bar{n}$ is the upper bound of n_k which satisfies $n_k = w_k - A_kK_kv_k$.*

Sketch of the Proof: We first construct a Lyapunov candidate for the estimated error vector ζ_k. Since Assumptions 4.1 and 4.2 are satisfied, it is known that the system parameters are bounded and the inequality discussed in Lemma 4.1 holds. By applying the bounds of system parameters and the inequality (16), we can prove that when $||\zeta_k|| > \varepsilon_1$, the Lyapunov candidate is negative. Detailed proof could be found in Appendix 6.2.

Without loss of generality, it should be noted that our proposed estimator also works for a class of systems that is not in the form of Luenberger second observable canonical form. If we can find a state space transformation $x_k = T^o x_k^o$ which brings the original system to the Luenberger second observable canonical form, then our estimator still works for the original system. If we cannot find such a state space transformation, from the analysis in Appendix 6.1 it is known that our analysis requires the matrix $A_k(I - K_kC_k)$ to be invertible. If the original system satisfies this condition, our analysis still holds and thus the estimator works for it.

4.2. The effect of data loss duration to the maximum estimation error

Theorem 4.1 gives the convergence condition of algorithm 1 if the observations are completely available. On the other hand, the convergence and the estimation error of proposed filtering algorithm are obviously influenced by the missing observations. In this subsection, we investigate the relationship between the maximum estimation error of filtering algorithm 1 and the duration of observation loss. In the following, $\varepsilon_{\max}$ is used to denote the maximum estimation error which is tolerable in the application of algorithm 1. n is used to denote the number of continuous missing observations, i.e. the observations are lost from time $k+1$ to $k+n$ with $n \ge 1$ for some $k \in \mathbb{N}$.

Theorem 4.2. *If the system state estimation error of the filtering algorithm 1 during the observation loss is bounded by $\|\zeta_{k+i}\| \le \varepsilon_{\max}$, where $i = (1, \cdots, n), k \ge 0$, then the number of continuous missing observations has the following properties:*

1. *If $\bar{a} = 1$, then $n \le \frac{\varepsilon_{\max} - ||\zeta_k||}{\bar{w}}$.*
2. *If $\bar{a} > 1$, then we have*

$$n \le \frac{\ln[(\bar{a} - 1)\varepsilon_{\max} + \bar{w}] - \ln[(\bar{a} - 1) \cdot \|\zeta_k\| + \bar{w}]}{\ln \bar{a}}. \tag{17}$$

3. *If $\bar{a} < 1$ and $(\bar{a}-1)\cdot\varepsilon_{\max} + \bar{w} > 0$, then we have (17).*
4. *If $\bar{a} < 1$ and $(\bar{a}-1)\cdot\varepsilon_{\max} + \bar{w} \le 0$, then the missing observations do not affect the estimation.*

Sketch of the Proof: Assuming that the observations are lost from time $k+1$ to $k+n$ with $n \ge 1$ for some $k \in \mathbb{N}$, the estimation error ζ_k may increase as time goes on during this period. We first need to derive the relationship between ζ_k and ζ_{k+n}. For all the four cases of this theorem, we can obtain the corresponding maximum bounds that ζ_{k+n} may reach. If the maximum tolerable estimation error $\varepsilon_{\max}$ is given, by solving the equality $\zeta_{k+n} \le \varepsilon_{\max}$ we can obtain the maximum tolerable number of continuous missing observations. Detailed proof could be found in Appendix 6.3.

Theorem 4.2 has revealed the relationship between the desired maximum estimation error and the allowable duration of observation loss. It should be noted that the accurate estimation error ζ_k is not known in the filtering process. What we know is only the updated ellipsoid which contains the true state value. For the practicability, we propose the following corollary.

Corollary 4.1. *Assume that all the assumptions in Theorem 4.2 hold, the estimation error of Algorithm 1 is bounded by $\varepsilon_{\max}$ if the following conditions are satisfied:*

1. *If $\bar{a} = 1$, then $n \le \frac{\varepsilon_{\max} - ||\Omega_{k|k-1}||_{\max}}{\bar{w}}$.*
2. *If $\bar{a} > 1$, then*

$$n \le \frac{\ln[(\bar{a}-1)\varepsilon_{\max} + \bar{w}] - \ln[(\bar{a}-1)\cdot||\Omega_{k|k-1}||_{\max} + \bar{w}]}{\ln \bar{a}}. \tag{18}$$

3. *If $\bar{a} < 1$ and $(\bar{a}-1)\cdot\varepsilon_{\max} + \bar{w} > 0$, then we have (18).*
4. *If $\bar{a} < 1$ and $(\bar{a}-1)\cdot\varepsilon_{\max} + \bar{w} \le 0$, then the missing observations do not affect the estimation.*

Here $||\Omega_{k|k-1}||_{\max} = \sigma_{k|k-1}\sqrt{\lambda_{\max}(P_{k|k-1})}$.

Sketch of the Proof: The proof generally follows the line of Theorem 4.2. We replace the role of estimation error ζ_k with the worst case estimation error for predicted ellipsoid $\Omega_{k|k-1}$. Detailed proof could be found in Appendix 6.4.

The corollary shows that if there is only limited observation loss during time $k+1$ to $k+n$, the estimation error can then be guaranteed to be smaller than the maximum estimation error $\varepsilon_{\max}$. Since $||\Omega_{k|k-1}||_{\max}$ can be obtained in each period, this corollary may be used for monitoring the proposed filtering algorithm online.

4.3. Estimation error tightened based on constraint observation dropout rate

From Theorem 4.1, we know that if the observations are available all the time, the estimation error will converge to a bound ε_1. Theorem 4.2 gives the relationship between the duration of data dropout and the maximum bound of estimation error. Based on the conditions of Theorem 4.2, actually, we can further reduce the estimation error if the data dropout rate is small enough. To explain this, we first introduce a result coming from the analysis of Theorem 4.1 in reference [2], we represent it as a lemma here.

Lemma 4.2. *If the following assumptions are satisfied:*

$$a_1 I_n \le \sum_{i=k}^{k+m-1} \tilde{A}_{k+m,i+1} Q_i \tilde{A}^T_{k+m,i+1} \le a_2 I_n,$$

$$b_1 I_n \le \sum_{i=k}^{k+s_k(m)} \tilde{A}^T_{i,k+s_{k(m)}} C_i^T R_i C_i \tilde{A}_{i,k+s_{k(m)}} \le b_2 I_n,$$

where $\tilde{A}_{k+j,k} = \tilde{A}_{k+j,k+j-1}\tilde{A}_{k+j-1,k+j-2}\cdots\tilde{A}_{k+1,k}$, $\tilde{A}_{k+1,k} = A_k$, $\tilde{A}_{k+j,k} = \tilde{A}^{-1}_{k,k+j}$, $\tilde{A}_{k,k} = I_n$, $s_k(m)$ (respectively $s_k(m) - m$) is the length of the sequence Υ_k among which m is the number of $\Upsilon_k = 1$ (respectively $\Upsilon_k = 0$). Then the matrices $P_{k+1|k}$ and P_k are bounded and given by $\underline{s_0}\cdot I \le P_{k+1|k} \le \bar{s}_0 \cdot I$, $\underline{s_0}\cdot I \le P_k \le \bar{s}_0\cdot I$.

The following definition is introduced to quantitatively describe the data dropout rate which is proposed in our previous work [13].

Definition 4.1. Let $r(N)$: $\{1, 2, \cdots\} \to [0, 1]$, then $r(N)$ is said to be the data dropout rate over N steps (DRNS) of system (1), (2), if $\forall N \in \{1, 2, \cdots\}$, $r(N)$ satisfies

$$r(N) = \sum_{k=1}^{N} \frac{1-\Upsilon_k}{N}.$$

With limited missing observations period and data dropout rate, we can obtain an tighter bound than the bound $\varepsilon_{\max}$ which is discussed in Theorem 4.2. Then we would like to introduce the following result.

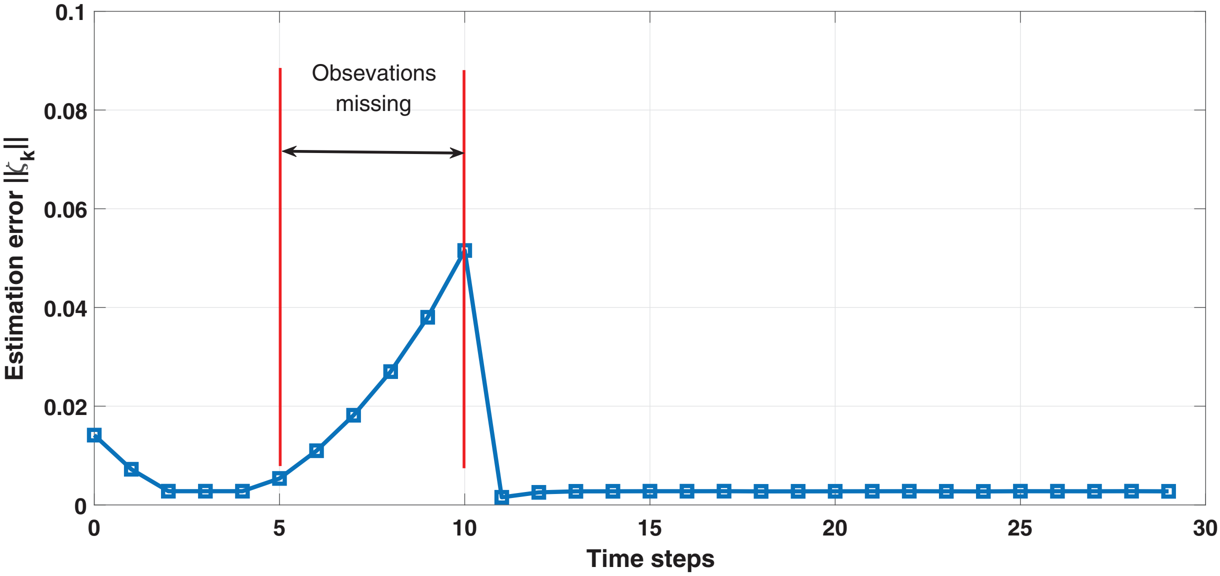

Fig. 3. Estimation error $||\zeta(k)||$ during observations missing period.

Theorem 4.3. *Suppose that Assumptions 4.1, 4.2 and all conditions in Theorem 4.2 and Lemma 4.2 are satisfied. Selecting an estimation error bound* ε_2 *which satisfies* $\frac{-\kappa_{30}-\sqrt{\kappa_{30}^2+4\kappa_{10}\kappa_{20}}}{-2\kappa_{20}} < \varepsilon_2 \le \varepsilon_{\max}$, *then the estimation error's bound will be tightened from* $\varepsilon_{\max}$ *to* ε_2 *if the following inequality satisfies:*

$$r(N) \le \frac{\underline{s}_0\left(\kappa_{20}\varepsilon_2^2 - \kappa_{30}\varepsilon_2 - \kappa_{10}\right)}{\left(2\bar{a}\bar{w}\varepsilon_{\max} + \bar{w}^2\right) + \underline{s}_0\left(\kappa_{20}\varepsilon_2^2 - \kappa_{30}\varepsilon_2 - \kappa_{10}\right)}, \tag{19}$$

where

$$\kappa_{10} = \frac{\bar{n}^2}{\underline{s}_0}, \kappa_{20} = \frac{(1-\alpha_k{}^{-1})}{\underline{s}_0} + \frac{1}{\bar{\alpha}\bar{s}_0^2(\underline{s}_0{}^{-1} + \alpha_k \bar{k}^{*2}\bar{a}^2/\underline{q})}, \kappa_{30} = \frac{2\bar{a}\bar{n}\bar{k}^*}{\underline{s}_0}.$$

Sketch of the Proof: The sketch for the proof of Theorem 4.3 can be explained as follows. We consider the whole system: either observations are available or no observations are available. In the first case the system is stable and we have a decreasing Lyapunov candidate like $\Delta V_{k,1} \le -c_1$, where c_1 is a positive constant. In the second case, the system is unstable. We have an increasing Lyapunov candidate like $\Delta V_{k,0} \ge c_2$, where c_2 is a positive constant. If the data dropout rate is limited so that it is possible to find a decreasing overall Lyapunov candidate $\Delta V_{k,1} + \Delta V_{k,0} < 0$, then we can say that the whole system is stable. The detailed proof can be found in Appendix 6.5.

5. Numerical examples

Consider the following discrete time-varying system with incomplete observation

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & 0.4+\alpha_k \\ 1 & 0.9 \end{bmatrix} x_k + w_k \\ y_k &= \Upsilon_k(\begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k). \end{aligned}$$

Note that this system is observable and (A_k, C_k) follows the Luenberger second observable canonical form. In this system, we have $\bar{a} > 1$. The state is $x_k = [\begin{smallmatrix} x_{1,k} \\ x_{2,k} \end{smallmatrix}]$ and α_k belongs to the known range $[\alpha_{\min}, \alpha_{\max}]$.

In all simulations, we consider that the system works in a finite-time interval of 50 samples. The initial state of this simulation is: $x_0 = [\begin{smallmatrix} 0.4 \\ 0.4 \end{smallmatrix}]$, $\hat{x}_0 = [\begin{smallmatrix} 0.41 \\ 0.39 \end{smallmatrix}]$, $P_0 = [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]$, $\sigma_0 = 0.01$. The other system parameters are $w_k = [\begin{smallmatrix} 0.003 \\ 0.003 \end{smallmatrix}]$, $Q_k = [\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]$, $v_k = 0.01$, $\alpha_k \in [0, 0.05]$.

An application of the proposed set-membership filter with incomplete observation has been reported in our previous work [36]. Indoor localization could be realized by data from wearable sensors and the RFID reading subsystem which

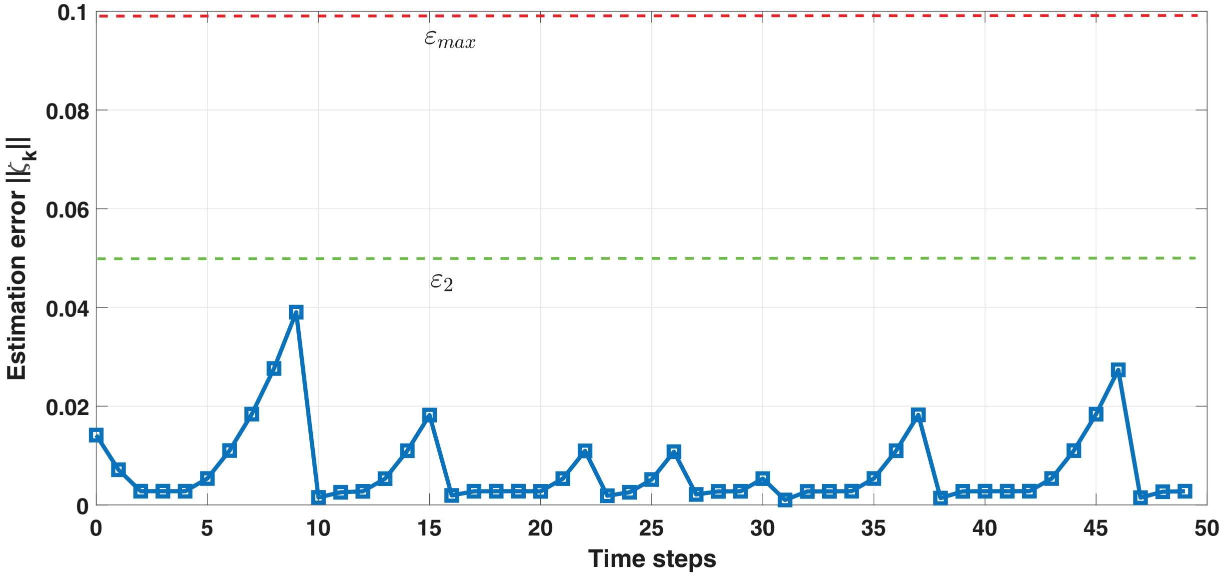

Fig. 4. Estimation error $||\zeta(k)||$ dynamics with $r(N) = 0.4$.

provides the observations. Note that the RFID tags were distributed at separate areas so that the observations are not guaranteed to be obtained at each time. Experimental results show that the proposed set-membership filter can significantly reduce the accumulated error of indoor localization system.

5.1. Checking the effectiveness of Theorems 4.2 and 4.3

First, we conducted a simulation to verify the effectiveness of Theorem 4.2. In this case, the prior bound of maximum estimation error is set as $\varepsilon_{\max} = 0.1$. According to Theorem 4.2, we can calculate that $n \le 6.41$ at time $k = 5$. Assume that the measured signal y_k is missing from time $k = 6$ to $k = 11$, the estimation error $||\zeta(k)||$ versus time is represented in Fig. 3. It is shown that the estimation error will not surpass $\varepsilon_{\max}$ if the duration of data loss satisfies the constraint given by Theorem 4.2. The curve also shows that the initial estimation error will decrease to a certain level if there are no missing observations.

Secondly, a simulation was conducted to verify the results of Theorem 4.3. In this case, according to Theorem 4.1, we can obtain $\varepsilon_1 = 0.005$. Choosing $\varepsilon_2 = 0.05$, then according to Theorem 4.3 we can calculate the DRNS of this system. It should satisfy $r(N) < 0.47$. Selecting the DRNS of this system as $r(N) = 0.4$, $m = 30$ and $s_k(30) = 50$ satisfy the assumptions in Lemma 4.2. Fig. 4 shows the dynamics of estimation error using the proposed filtering algorithm.

From Fig. 4 we can easily find that if the conditions in Theorem 4.3 are satisfied, we can tighten the estimation bound from $\varepsilon_{\max}$ to ε_2, which verifies the effectiveness of Theorem 4.3.

5.2. Comparison with the Kalman interval filter

This section is devoted to the comparison of our algorithm with the Interval Kalman Filter proposed in [33]. The Interval Kalman Filter of [33] has not been designed to deal with missing observations, so the idea of this comparison is to show that our algorithm can provide better results when there are such missing observations. Note that in [33], an interval Kalman filter is introduced to deal with the parameter uncertainties, whereas we do not consider parameter uncertainties here so that the related intervals are taken as exact system parameters. We use the same UBB noise and apply the predicted value as the final estimated value when observations are missing from step 11 to 16, and step 31 to 33. The comparison result is shown in Fig. 5. From this plot we can see that the proposed set-membership filter achieved better performance in the case of UBB noise, which shows that our algorithm for dealing with incomplete observations worked.

6. Conclusions

This paper deals with the set-membership filtering problem with incomplete observation for discrete time-varying systems. The relationship between missing observations and estimation error is discussed and a recursive algorithm has been proposed to calculate an ellipsoid which always contains the true value. The algorithm is very suitable for online applications with missing observations such as networked control system. In the future, we also would like to extend our method to more complicated systems, e.g., the switched or hybrid systems.

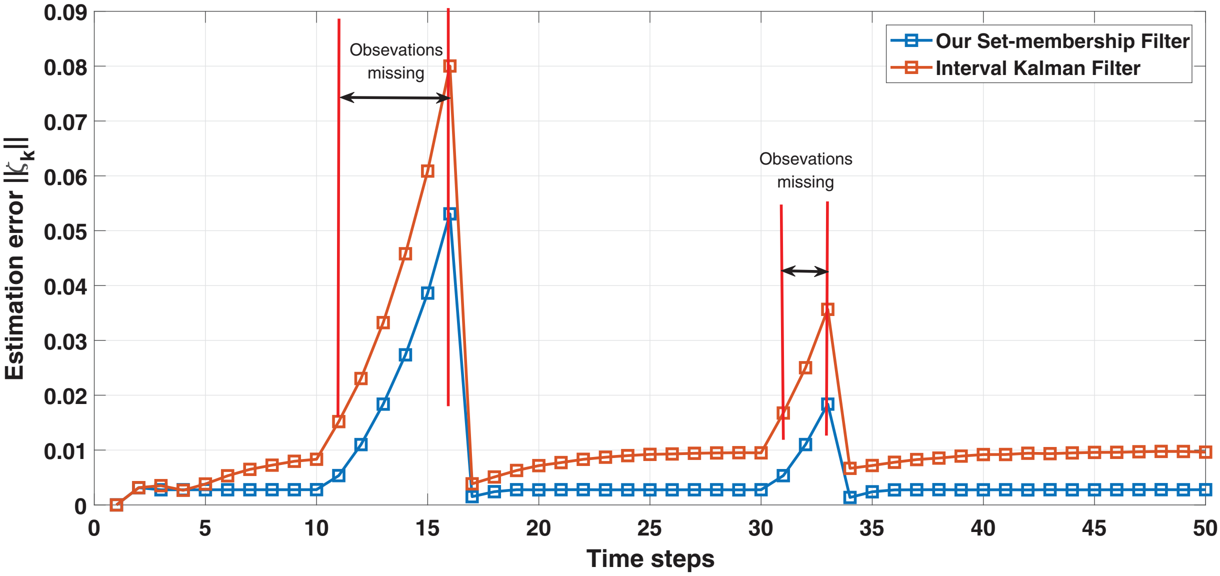

Fig. 5. Comparison with the Interval Kalman Filter of [33].

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Yuan Wang: Investigation, Writing - original draft. **Jian Huang:** Conceptualization, Methodology. **Dongrui Wu:** Validation. **Zhi-Hong Guan:** Writing - review & editing. **Yan-Wu Wang:** Writing - review & editing.

Acknowledgment

This work was supported by the National Natural Science Foundation of China under grants U1913207, 61873321 and 81601348, the Hubei Technology Innovation Platform under grant 2019AEA171.

Appendix A

6.1. Proof of Lemma 4.1

We first show that $A_k(I - K_kC_k)$ is invertible. Write A_k, C_k and K_k in a partitioned form of Luenberger second observable canonical form, then we have

$$A_k = \begin{bmatrix} \mathbf{0_a} & a_1 \\ I_{n-1} & \mathbf{a}_{n-1} \end{bmatrix}, C_k = \begin{bmatrix} \mathbf{0_c} & \mathbf{1_c} \end{bmatrix}, K_k = \begin{bmatrix} \mathbf{k_1} \\ \mathbf{k} \end{bmatrix},$$

where zero vector $\mathbf{0_a} \in \mathbb{R}^{1\times(n_x-1)}$, identity matrix $I_{n-1} \in \mathbb{R}^{(n_x-1)\times(n_x-1)}$, zero matrices $\mathbf{0_c} \in \mathbb{R}^{n_y\times(n_x-1)}$, $\mathbf{1_c} \in \mathbb{R}^{n_y}$ is a vector with elements equal to 1. For matrices $K_k \in \mathbb{R}^{n_x\times n_y}$, $\mathbf{k_1} \in \mathbb{R}^{1\times n_y}$, $\mathbf{k} \in \mathbb{R}^{(n_x-1)\times n_y}$. Then we can write $A_k(I - K_kC_k)$ in partitioned form

$$A_k(I - K_kC_k) = \begin{bmatrix} \mathbf{0_a} & l_1 \\ I_{n-1} & \mathbf{l}_{n-1} \end{bmatrix}.$$

Obviously, $A_k(I - K_kC_k)$ is invertible.

Note that (13) can be written as

$$P_{k+1} = (I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T + \frac{1}{q_{k+1}}K_{k+1}K_{k+1}^T. \tag{20}$$

The reason is explained in the following.

From the notion of K_{k+1}, we know that

$$P_{k+1|k}C_{k+1}^T\left(\frac{1}{q_{k+1}}I+C_{k+1}P_{k+1|k}C_{k+1}^T\right)^{-1}=K_{k+1},$$

$$P_{k+1|k}C_{k+1}^T=K_{k+1}\cdot(C_{k+1}P_{k+1|k}C_{k+1}^T+\frac{1}{q_{k+1}}I),$$

$$P_{k+1|k}C_{k+1}^TK_{k+1}^T=K_{k+1}C_{k+1}P_{k+1|k}C_{k+1}^TK_{k+1}^T+\frac{1}{q_{k+1}}K_{k+1}K_{k+1}^T. \tag{21}$$

It follows from (21) that

$$(I-K_{k+1}C_{k+1})P_{k+1|k}=(I-K_{k+1}C_{k+1})P_{k+1|k}(I-K_{k+1}C_{k+1})^T+\frac{1}{q_{k+1}}K_{k+1}K_{k+1}^T.$$

Thus we can obtain (20).

From (20) it follows that

$$P_{k+1}\geq(I-K_{k+1}C_{k+1})P_{k+1|k}(I-K_{k+1}C_{k+1})^T. \tag{22}$$

From (12) we have

$$P_{k+1|k}^{-1}=(\alpha_kA_kP_kA_k^T+Q_k^*)^{-1}. \tag{23}$$

Substitute (22) into (23), then we have

$$P_{k+1|k}^{-1}\leq([\alpha_k^{-1}A_k^{-T}(I-K_kC_k)^{-T}P_{k|k-1}^{-1}(I-K_kC_k)^{-1}A_k^{-1}]^{-1}+Q_k^*)^{-1}. \tag{24}$$

By applying (15) into (24), it follows that

$$\begin{aligned}P_{k+1|k}^{-1}&\leq\alpha_k^{-1}A_k^{-T}(I-K_kC_k)^{-T}P_{k|k-1}^{-1}(I-K_kC_k)^{-1}A_k^{-1}-\alpha_k^{-1}A_k^{-T}(I-K_kC_k)^{-T}P_{k|k-1}^{-1}(I-K_kC_k)^{-1}A_k^{-1}\cdot\\&\quad\left[\alpha_k^{-1}A_k^{-T}(I-K_kC_k)^{-T}P_{k|k-1}^{-1}(I-K_kC_k)^{-1}A_k^{-1}+Q_k^{*-1}\right]^{-1}\cdot\\&\quad\alpha_k^{-1}A_k^{-T}(I-K_kC_k)^{-T}P_{k|k-1}^{-1}(I-K_kC_k)^{-1}A_k^{-1}\\&=\alpha_k^{-1}A_k^{-T}(I-K_kC_k)^{-T}P_{k|k-1}^{-1}(I-K_kC_k)^{-1}A_k^{-1}-\alpha_k^{-1}A_k^{-T}(I-K_kC_k)^{-T}P_{k|k-1}^{-1}\cdot\\&\quad\left[P_{k|k-1}^{-1}+\alpha_k(I-K_kC_k)^TA_k^TQ_k^{*-1}A_k(I-K_kC_k)\right]^{-1}\cdot\\&\quad P_{k|k-1}^{-1}(I-K_kC_k)^{-1}A_k^{-1}=\alpha_k^{-1}A_k^{-T}(I-K_kC_k)^{-T}\cdot\\&\quad\left[P_{k|k-1}^{-1}-P_{k|k-1}^{-1}\left[P_{k|k-1}^{-1}+\alpha_k(I-K_kC_k)^TA_k^TQ_k^{*-1}A_k(I-K_kC_k)\right]^{-1}P_{k|k-1}^{-1}\right]\cdot\\&\quad(I-K_kC_k)^{-1}A_k^{-1}.\end{aligned}$$

This completes the proof of lemma 4.1. □

6.2. proof of Theorem 4.1

Because there is no missing observation, it follows that

$$\begin{aligned}\zeta_{k+1}&=x_{k+1}-x_{k+1|k}=A_kx_k+w_k-A_k\hat{x}_k\\&=A_kx_k+w_k-A_k\left(x_{k|k-1}+K_k(y_k-C_kx_{k|k-1})\right)\\&=A_kx_k+A_kK_kC_kx_k-A_kK_kC_kx_k+w_k-A_kx_{k|k-1}-A_kK_ky_k+A_kK_kC_kx_{k|k-1}\\&=A_k(x_k-x_{k|k-1})-A_kK_kC_k(x_k-x_{k|k-1})+A_kK_kC_kx_k+w_k-A_kK_ky_k\\&=A_k(I-K_kC_k)\zeta_k+A_kK_kC_kx_k+w_k-A_kK_ky_k.\end{aligned} \tag{25}$$

Substitute (2) into (25), we have

$$\zeta_{k+1}=A_k(I-K_kC_k)\zeta_k-A_kK_kv_k+w_k.$$

Let the Lyapunov candidate be $V_{k+1}(\zeta_{k+1})=\zeta_{k+1}^TP_{k+1|k}^{-1}\zeta_{k+1}$. Then we have

$$\begin{aligned}V_{k+1}(\zeta_{k+1})=\zeta_{k+1}^TP_{k+1|k}^{-1}\zeta_{k+1}&=(A_k(I-K_kC_k)\zeta_k+n_k)^TP_{k+1|k}^{-1}\cdot(A_k(I-K_kC_k)\zeta_k+n_k)\\&=\zeta_k^T(I-K_kC_k)^TA_k^TP_{k+1|k}^{-1}A_k(I-K_kC_k)\zeta_k+\zeta_k^T(I-K_kC_k)^TA_k^TP_{k+1|k}^{-1}n_k\\&\quad+n_k^TP_{k+1|k}^{-1}A_k(I-K_kC_k)\zeta_k+n_k^TP_{k+1|k}^{-1}n_k.\end{aligned} \tag{26}$$

By applying Lemma 4.1 in (26), we have

$$\begin{aligned} V_{k+1}(\zeta_{k+1}) \le\ & \zeta_k^T (I-K_kC_k)^T A_k^T \{\alpha_k^{-1} A_k^{-T} (I-K_kC_k)^{-T} \cdot \\ & [P_{k|k-1}^{-1} - P_{k|k-1}^{-1} (P_{k|k-1}^{-1} + \alpha_k (I-K_kC_k)^T A_k^T Q_k^{*-1} A_k (I-K_kC_k))^{-1} P_{k|k-1}^{-1}] \cdot \\ & (I-K_kC_k)^{-1} A_k^{-1}\} A_k (I-K_kC_k)\zeta_k + \zeta_k^T (I-K_kC_k)^T A_k^T P_{k+1|k}^{-1} n_k + n_k^T P_{k+1|k}^{-1} A_k (I-K_kC_k)\zeta_k + n_k^T P_{k+1|k}^{-1} n_k \\ =\ & \alpha_k^{-1} \zeta_k^T [P_{k|k-1}^{-1} - P_{k|k-1}^{-1} (P_{k|k-1}^{-1} + \alpha_k (I-K_kC_k)^T A_k^T Q_k^{*-1} A_k (I-K_kC_k))^{-1} \cdot \\ & P_{k|k-1}^{-1}]\zeta_k + \zeta_k^T (I-K_kC_k)^T A_k^T P_{k+1|k}^{-1} n_k + n_k^T P_{k+1|k}^{-1} A_k (I-K_kC_k)\zeta_k + n_k^T P_{k+1|k}^{-1} n_k. \end{aligned} \tag{27}$$

From Eq. (13), we know that $I-K_kC_k$ is positive definite matrix. According to Assumptions 4.1 and 4.2, the matrices $I-K_kC_k$ is bounded. Using the appropriate upper and lower bounds in (27), we have

$$V_{k+1}(\zeta_{k+1}) \le \alpha_k^{-1} \zeta_k^T P_{k|k-1}^{-1} \zeta_k - \frac{1}{\alpha_k \bar{s}^2 (\underline{s}^{-1} + \alpha_k \bar{k}^{*2} \bar{a}^2 / \underline{q})} ||\zeta_k||^2 + \frac{2\bar{a}\bar{n}\bar{k}^*}{\underline{s}} ||\zeta_k|| + \frac{\bar{n}^2}{\underline{s}}.$$

This results in

$$V_{k+1}(\zeta_{k+1}) - V_k(\zeta_k) \le (\alpha_k^{-1} - 1) V_k(\zeta_k) - \frac{1}{\alpha_k \bar{s}^2 (\underline{s}^{-1} + \alpha_k \bar{k}^{*2} \bar{a}^2 / \underline{q})} ||\zeta_k||^2 + \frac{2\bar{a}\bar{n}\bar{k}^*}{\underline{s}} ||\zeta_k|| + \frac{\bar{n}^2}{\underline{s}}.$$

Considering the bound of Lyapunov candidate, we have

$$\frac{1}{\bar{s}} ||\zeta_k||^2 \le V_k(\zeta_k) \le \frac{1}{\underline{s}} ||\zeta_k||^2.$$

It follows that

$$V_{k+1}(\zeta_{k+1}) - V_k(\zeta_k) \le \frac{1}{\bar{s}} (\alpha_k^{-1} - 1) ||\zeta_k||^2 - \frac{1}{\alpha_k \bar{s}^2 (\underline{s}^{-1} + \alpha_k \bar{k}^{*2} \bar{a}^2 / \underline{q})} ||\zeta_k||^2 + \frac{2\bar{a}\bar{n}\bar{k}^*}{\underline{s}} ||\zeta_k|| + \frac{\bar{n}^2}{\underline{s}}.$$

By applying the notion of κ_1, κ_2 and κ_3, we have

$$V_{k+1}(\zeta_{k+1}) - V_k(\zeta_k) \le -\kappa_2 ||\zeta_k||^2 + \kappa_3 ||\zeta_k|| + \kappa_1. \tag{28}$$

It is easy to see that $V_{k+1}(\zeta_{k+1}) - V_k(\zeta_k) < 0$ will hold for all $||\zeta_k||$ satisfying $||\zeta_k|| \ge \frac{-\kappa_3 - \sqrt{\kappa_3^2 + 4\kappa_1\kappa_2}}{-2\kappa_2}$. This completes the proof. □

6.3. proof of Theorem 4.2

In the case of missing observation, we have $\hat{x}_{k+1} = x_{k+1|k}$ and $P_{k+1} = P_{k+1|k}$. Thus the system estimation error in this case is represented as

$$\begin{aligned} \zeta_{k+1} &= x_{k+2} - x_{k+2|k+1} \\ &= A_{k+1} x_{k+1} + w_{k+1} - A_{k+1} \hat{x}_{k+1} \\ &= A_{k+1} x_{k+1} + w_{k+1} - A_{k+1} x_{k+1|k} \\ &= A_{k+1} (x_{k+1} - x_{k+1|k}) + w_{k+1}. \end{aligned}$$

Obviously the dynamics of systems estimation error is:

$$\zeta_{k+1} = A_{k+1} \zeta_k + w_{k+1}.$$

After n steps, we can obtain the estimation error ζ_{k+n}. Considering that the upper bound of w_k is $||\bar{w}||$, we have

$$\begin{aligned} ||\zeta_{k+n}|| &= ||A_{k+n-1} \zeta_{k+n-1} + w_{k+n-1}|| \\ &= ||A_{k+n-1} [A_{k+n-2} \zeta_{k+n-2} + w_{k+n-2}] + w_{k+n-1}|| \\ &= \cdots \\ &= ||A_{k+n-1} A_{k+n-2} \ldots A_k \zeta_k + w_{k+n-1} + A_{k+n-1} w_{k+n-2} \\ &\quad + A_{k+n-1} A_{k+n-2} w_{k+n-3} + \ldots + A_{k+n-1} A_{k+n-2} \ldots A_{k+1} w_k|| \\ &\le ||\prod_{i=0}^{n-1} A_{k+i} \zeta_k|| + \bar{w} + ||A_{k+n-1}|| \cdot \bar{w} + \cdots + ||A_{k+n-1} A_{k+n-2} \ldots A_{k+1}|| \\ &= ||\prod_{i=0}^{n-1} A_{k+i} \zeta_k|| + \bar{w} + \sum_{i=1}^{n-1} \prod_{j=1}^{i} ||A_{k+n-j}|| \cdot \bar{w}. \end{aligned}$$

Thus we have

$$||\zeta_{k+n}|| \leq \prod_{i=0}^{n-1} ||A_{k+i}|| \cdot ||\zeta_k|| + \bar{w} + \sum_{i=1}^{n-1} \prod_{j=1}^{i} ||A_{k+n-j}|| \cdot \bar{w}.$$

If $\bar{a} = 1$, it follows that

$$||\zeta_{k+n}|| \leq ||\zeta_k|| + n\bar{w},$$

$$||\zeta_k|| + n\bar{w} \leq \varepsilon_{\max}. \tag{29}$$

The second result of this theorem is then proved by the solution of (29).

If $\bar{a} \neq 1$, it follows that

$$\begin{aligned} ||\zeta_{k+n}|| &\leq \bar{a}^n \cdot ||\zeta_k|| + \bar{w} + \sum_{i=1}^{n-1} \bar{a}^i \cdot \bar{w} \\ &= \bar{a}^n \cdot ||\zeta_k|| + \frac{1-\bar{a}^n}{1-\bar{a}} \cdot \bar{w}. \end{aligned}$$

Since $||\zeta_{k+i}|| \leq \varepsilon_{\max}$ holds, we have

$$\bar{a}^n \cdot \parallel \zeta_k \parallel + \frac{1-\bar{a}^n}{1-\bar{a}} \cdot \bar{w} \leq \varepsilon_{\max},$$

$$\bar{a}^n \cdot ||\zeta_k|| - \frac{\bar{w}}{1-\bar{a}} \cdot \bar{a}^n \leq \varepsilon_{\max} - \frac{\bar{w}}{1-\bar{a}},$$

$$\bar{a}^n \cdot (||\zeta_k|| + \frac{\bar{w}}{\bar{a}-1}) \leq \varepsilon_{\max} + \frac{\bar{w}}{\bar{a}-1}. \tag{30}$$

If $\bar{a} > 1$, it follows that

$$\bar{a}^n \cdot [(\bar{a}-1) \cdot ||\zeta_k|| + \bar{w}] \leq (\bar{a}-1)\varepsilon_{\max} + \bar{w},$$

$$\bar{a}^n \leq \frac{(\bar{a}-1)\varepsilon_{\max} + \bar{w}}{(\bar{a}-1) \cdot ||\zeta_k|| + \bar{w}}. \tag{31}$$

Taking the logarithm of both sides of (31), then we have:

$$n \cdot \ln \bar{a} \leq \ln[(\bar{a}-1)\varepsilon_{\max} + \bar{w}] - \ln[(\bar{a}-1) \cdot ||\zeta_k|| + \bar{w}]. \tag{32}$$

Part of the first result of this theorem is then proved by the solution of (32).

If $\bar{a} < 1$, from (30) it follows that

$$\bar{a}^n \cdot [(\bar{a}-1) \cdot ||\zeta_k|| + \bar{w}] \geq (\bar{a}-1)\varepsilon_{\max} + \bar{w}.$$

If $(\bar{a}-1)\varepsilon_{\max} + \bar{w} > 0$, due to $||\zeta_k|| \leq \varepsilon_{\max}$ we have $(\bar{a}-1) \cdot ||\zeta_k|| + \bar{w} > 0$. This results in

$$\bar{a}^n \geq \frac{(\bar{a}-1)\varepsilon_{\max} + \bar{w}}{(\bar{a}-1) \cdot ||\zeta_k|| + \bar{w}}. \tag{33}$$

Taking the logarithm of both sides of (33), we have

$$n \cdot \ln \bar{a} \geq \ln[(\bar{a}-1)\varepsilon_{\max} + \bar{w}] - \ln[(\bar{a}-1) \cdot \|\zeta_k\| + \bar{w}]. \tag{34}$$

Because $\ln \bar{a} < 0$ holds, another part of the first result of this theorem can be proved by the solution of Eq. (34). When $(\bar{a}-1)\varepsilon_0 + \bar{w} \leq 0$, it is obvious that the following inequality always holds

$$\bar{a}^n \cdot [(\bar{a}-1) \cdot ||\zeta_k|| + \bar{w}] \geq \bar{a}^n \cdot [(\bar{a}-1) \cdot \varepsilon_{\max} + \bar{w}] \geq (\bar{a}-1) \cdot \varepsilon_{\max} + \bar{w}.$$

This completes the proof. □

6.4. proof of Corollary 4.1

From the definition of ζ_k we know that ζ_k is the geometric distance between x_k and $x_{k|k-1}$. It is easy to see that ζ_k is smaller than the worst-case estimation error which is defined in definition 7, so we have

$$||\zeta_k|| \le ||\Omega_{k|k-1}||_{\max}$$

Then we can easily obtain

$$\begin{aligned}&\frac{\ln[(\bar{a}-1)\varepsilon_{\max}+\bar{w}]-\ln\left[(\bar{a}-1)\cdot\left\|\Omega_{k|k-1}\right\|_{\max}+\bar{w}\right]}{\ln\bar{a}}\\ \le&\frac{\ln[(\bar{a}-1)\varepsilon_{\max}+\bar{w}]-\ln[(\bar{a}-1)\cdot\|\zeta_k\|+\bar{w}]}{\ln\bar{a}}.\end{aligned}$$

The rest of proof is similar to that of Theorem 4.2. □

6.5. proof of Theorem 4.3

At those time instants when the observations are lost, we have $I-K_kC_k=I$ and $n_k=w_k$. Thus, we can obtain the following Lyapunov candidate from (26):

$$V_{k+1}(\zeta_{k+1}) = \zeta_k^T A_k^T P_{k+1|k}^{-1} A_k\zeta_k + 2w_k^T P_{k+1|k}^{-1} A_k\zeta_k + w_k^T P_{k+1|k}^{-1} w_k$$

Note that Lemma 4.1 still holds but $I-K_kC_k=I$ in this case. By applying Lemma 4.1, we have

$$\begin{aligned}V_{k+1}(\zeta_{k+1}) \le&\alpha_k^{-1}\zeta_k^T[P_{k|k-1}^{-1}-P_{k|k-1}^{-1}(P_{k|k}^{-1}+\alpha_k A_k^T Q_k^{*-1}A_k)^{-1}P_{k|k-1}^{-1}]\zeta_k\\ &+2w_k^T P_{k+1|k}^{-1}A_k\zeta_k + w_k^T P_{k+1|k}^{-1}w_k.\end{aligned}$$

Using the appreciate upper and lower bound, it follows that

$$V_{k+1}(\zeta_{k+1})-V_k(\zeta_k) \le \left(\alpha_k^{-1}-1\right)V_k(\zeta_k) - \frac{1}{\alpha_k\bar{s}_0^2\left(\underline{s_0}^{-1}+\alpha_k\bar{a}^2/q\right)}||\zeta_k||^2 + \frac{2\bar{a}\bar{w}}{\underline{s_0}}||\zeta_k|| + \frac{\bar{w}^2}{\underline{s_0}}. \tag{35}$$

Because inequality $\alpha_k{}^{-1}-1<0$ always holds, then we have

$$V_{k+1}(\zeta_{k+1})-V_k(\zeta_k) \le \frac{2\bar{a}\bar{w}}{\underline{s_0}}||\zeta_k|| + \frac{\bar{w}^2}{\underline{s_0}} \le \frac{1}{\underline{s_0}}\left(2\bar{a}\bar{w}\varepsilon_{\max}+\bar{w}^2\right).$$

At the time instants of no data missing, the similar conclusion to Theorem 4.1 can be obtained using the new upper and lower bounds of P_k in Lemma 4.2. Meanwhile the notation of parameters κ_1, κ_2, κ_3 should be changed to κ_{10}, κ_{20}, κ_{30}. From (28) we know that

$$V_{k+1}(\zeta_{k+1})-V_k(\zeta_k) \le -\kappa_{20}||\zeta_k||^2+\kappa_{30}||\zeta_k||+\kappa_{10}. \tag{36}$$

If $\frac{-\kappa_{30}-\sqrt{\kappa_{30}^2+4\kappa_{10}\kappa_{20}}}{-2\kappa_{20}} < \varepsilon_2 \le ||\zeta_k|| < \varepsilon_{\max}$, we know that $-\kappa_{20}||\zeta_k||^2+\kappa_{30}||\zeta_k||+\kappa_{10}<0$.

Let $f(||\zeta_k||) = -\kappa_{20}||\zeta_k||^2+\kappa_{30}||\zeta_k||+\kappa_{10}$, then depending on the monotonic of $f(||\zeta_k||)$, in the area of $\varepsilon_2 \le ||\zeta_k|| < \varepsilon_{\max}$, the maximum value of $f(||\zeta_k||)$ is

$$f(\varepsilon_2) = -\kappa_{20}\varepsilon_2^2+\kappa_{30}\varepsilon_2+\kappa_{10}<0.$$

Let ΔV_0 be the one-step increment of Lyapunov candidate at the time instants of missing data, and ΔV_1 the one-step decrement of Lyapunov candidate at the time instants of no missing data. In the whole N steps, if

$$\sum_{i=1}^{N\cdot r(N)}\Delta V_{0,i} + \sum_{i=1}^{N-N\cdot r(N)}\Delta V_{1,i} < 0,$$

then we can say that the filter is stable.

In the period of missing observations, from (35) it follows that

$$\Delta V_0 \le \frac{1}{\underline{s_0}}(2\bar{a}\bar{w}\varepsilon_{max}+\bar{w}^2).$$

Then we have

$$\Delta V_1 \le -\kappa_{20}\varepsilon_2^2+\kappa_{30}\varepsilon_2+\kappa_{10}.$$

Thus,

$$\sum_{i=1}^{N\cdot r(N)}\Delta V_{0,i} + \sum_{i=1}^{N-N\cdot r(N)}\Delta V_{1,i} < Nr(N)\cdot\frac{1}{\underline{s_0}}\left(2\bar{a}\bar{w}\varepsilon_{\max}+\bar{w}^2\right) + (N-Nr(N))\cdot\left(-\kappa_{20}\varepsilon_2^2+\kappa_{30}\varepsilon_2+\kappa_{10}\right).$$

If inequality

$$Nr(N)\cdot\frac{1}{\underline{s_0}}\left(2\bar{a}\bar{w}\varepsilon_{\max}+\bar{w}^2\right)+(N-Nr(N))\cdot\left(-\kappa_{20}\varepsilon_2^2+\kappa_{30}\varepsilon_2+\kappa_{10}\right)\le 0 \quad (37)$$

is satisfied, then the filter is stable, the solution of (37) is (19).

Therefore if the estimation error is within $[\varepsilon_2, \varepsilon_{\max}]$, it will decrease and converge to ε_2. That is, the estimation bound is tighten from $\varepsilon_{\max}$ to ε_2. □

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