Fuzzy Optim Decis Making
https://doi.org/10.1007/s10700-020-09325-w

# Probabilistic linguistic multi-criteria decision-making based on double information under imperfect conditions

**Na Yue[1] · Dongrui Wu[2] · Jialiang Xie[1,3,4] · Shuili Chen[5]**



**Abstract** In this paper, we study four projection-based normalization models and a decision-making method for probabilistic linguistic multi-criteria decision-making problems, in which the assessment information about an alternative with respect to a criterion is incomplete and the criteria weight values are not precisely known but the ranges are available. To apply the projection to the probabilistic linguistic environment, we propose the equivalent expression forms of the probabilistic linguistic term sets, and then the equivalent transformation functions between the probabilistic linguistic term set and its associated vector are presented to realize the conversion between the operations on the probabilistic linguistic term sets and the operations on their associated vectors. Next, the projection formulas of the probabilistic linguistic term sets are introduced to build different normalization models for different types of uncertain probabilistic linguistic multi-criteria decision-making problems. After that,

✉ Jialiang Xie
xiejialiang@jmu.edu.cn

Na Yue
nayue163@163.com

Dongrui Wu
drwu@hust.edu.cn

Shuili Chen
sgzx@jmu.edu.cn

1 College of Science, Jimei University, Xiamen 361021, China

2 School of Artificial Intelligence and Automation, Huazhong University of Science and Technology, Wuhan 430074, China

3 Key Laboratory of Applied Mathematics of Fujian Province University, Putian 351100, China

4 Digital Fujian big data modeling and intelligent computing institute, Xiamen 361021, China

5 Chengyi Institute of Applied Technology, Jimei University, Xiamen 361021, China

Published online: 09 June 2020

a new deviation degree formula is proposed to account for the rationality and validity of the normalization models from the theoretical perspective. Finally, the probabilistic linguistic two-step method is used to determine the criteria weights values and rank the alternatives, and the validity of these projection-based normalization models and our proposed decision-making method are illustrated by a case about the performance assessment of data hiding techniques.

**Keywords** Probabilistic linguistic term set · Projection · Normalization model · Decision-making method · Data hiding

## 1 Introduction

Since the probabilistic linguistic term set (PLTS) was put forward by Pang et al. (2016) to preserve all the original linguistic information provided by the decision-makers (DMs), many achievements have been made on PLTSs. These results can be divided into three categories: (1) The basic operations for PLTSs, such as the operational laws (Pang et al. 2016; Gou and Xu 2016; Yue et al. 2020), distance measures (Lin et al. 2019; Lin and Xu 2018), possibility degree formulas (Feng et al. 2019), and probabilistic linguistic preference relations (Gao et al. 2019; Song and Hu 2019). (2) The extension of the PLTS, for example, the probabilistic uncertain linguistic term sets (Lin et al. 2018) and the interval-valued probabilistic linguistic term sets (Bai et al. 2018). (3) The methods for solving the probabilistic linguistic multi-criteria decision making (PL-MCDM) problems (Liao et al. 2017, 2019; Liu and Teng 2019; Wu et al. 2018). PLTSs have achieved very good results in applications such as modern medicine (Pan et al. 2018), edge computing (Lin et al. 2018), water security evaluation (Zhang et al. 2018b), etc.

In PL-MCDM problems, the DMs are asked to offer their assessment information in the form of PLTS. However, in the practical PL-MCDM problems, two uncertain situations may arise due to unfamiliar with the decision-making problems or other reasons. The first one is that the PLTS we obtained is incomplete, that is, the sum of the probabilities of all possible linguistic terms in a PLTS is less than 1. For instance, when the DMs assess the robustness of the LSB (Least Significant Bits) technique which is a highly professional decision-making problem, 50% of the DMs think that the robustness of the LSB technique is "rather poor", 20% of the DMs hold that it is "fairly good", and the others do not supply any assessment information due to the professionalism and the complexity of this problem. Thus, the assessment information we obtained is $\{rather\ poor(0.5),\ fairly\ good(0.2)\}$, which is a PLTS. And the PLTS we obtained is incomplete as the sum of the probabilities of all possible linguistic terms in the above PLTS is less than 1. The second one is that the criteria weights values are not precisely known but the ranges are available. Furthermore, a decision-making team can be composed of one DM or multiple DMs, thus the PL-MCDM problems based on double information under imperfect conditions can be divided into two types: the uncertain single DM PL-MCDM problems and the uncertain multiple DMs PL-MCDM problems. Commonly, these two uncertain PL-MCDM problems appear in our daily lives. Therefore, in this paper, we mainly investigate the methods to supplement the incomplete assessment information and determine the exact values of the criteria weights.

For the incomplete PLTS in the uncertain PL-MCDM problems, it needs to be normalized at the beginning of the decision-making processes. Up to now, two major normalization approaches have been proposed. The normalization method proposed by Pang et al. (2016) is the most commonly used in the uncertain PL-MCDM problems. This normalization method assigns the ignorance to all linguistic terms that are provided by DMs via using the simple arithmetic mean method but neglects the influence of the ignorance on the remaining linguistic terms in the linguistic term set (LTS) except those that appear in PLTS. For example, let $PL(p) = \{l_3(0.25), l_5(0.1)\}$ be a PLTS on LTS $\widetilde{L} = \{l_0, l_1, \ldots, l_6\}$, then we obtain the normalization of the PLTS $PL(p)$ is $PL^*(p) = \left\{l_3\left(\frac{0.25}{0.25+0.1}\right), l_5\left(\frac{0.1}{0.25+0.1}\right)\right\} = \left\{l_3(0.7), l_5(0.3)\right\}$ via using this normalization method. But from the above calculation process, we find that the possibilities of the linguistic terms $l_0, l_1, l_2, l_4$ and $l_6$ in LTS $\widetilde{L}$ that do not appear in PLTS $PL(p)$ are not considered in the normalization process. Therefore, the disadvantage of this normalization method (Pang et al. 2016) is that it does not take into account the possibilities of the remaining linguistic terms in LTS except those that appear in PLTS. Furthermore, through the combination rule of the evidence theory, Ma et al. (2018) constructed a nonlinear model to cope with the incomplete assessment information as another normalization method. But this normalization method is only suitable for the uncertain multiple DMs PL-MCDM problems. In short, these two existing normalization methods both have certain limitations, and the normalized results derived by these two normalization methods may deviate from the true opinions of the DMs.

For the criteria weights values that are not precisely known but the ranges are available, many classical methods have been proposed. Xu (2004) proposed a series of methods to determine the criteria weights values for the multi-criteria decision making (MCDM) problems, such as the maximum deviation method based on the deviation degree and the possibility degree (Xu 2001), the classical two-step method (Xu 2002), etc. Zhang et al. (2018b) developed a programming model to calculate the attributes weights for the water security issue under the probabilistic linguistic circumstance. Wu and Liao (2018) introduced an integrated weight-determining method for the uncertain PL-MCDM problems where the criterion weight information is completely unknown. Furthermore, for the uncertain PL-MCDM problems where the criterion weight information is completely unknown or partially known, Pang et al. (2016) proposed a single-objective optimization model based on the maximizing deviation method to determine the criteria weights. However, there is little research on the weight-determining method for the uncertain PL-MCDM problems where the criteria weights values that are not precisely known but the ranges are available.

To improve the accuracy of the normalization method, different types of the uncertain PL-MCDM problems should employ different ways to normalize their incomplete PLTSs. For example, for the uncertain single DM PL-MCDM problems, the normalization method of the incomplete PLTS should not only consider the possibilities of the remaining linguistic terms in LTS except those that appear in PLTS but also consider the individual similarity degree between the PLTS and its normalization as much as possible. But for the uncertain multiple DMs PL-MCDM problems, the group similarity degree is an important feature, so the normalization method of the incomplete PLTSs in these problems should consider the probabilities of the remaining linguistic terms in LTS and the group similarity degree simultaneously, then each normalized PLTS derived by this method will be closer to the true assessment information and can reduce the outliers to some extent. Therefore, in consideration of the possibilities of the remaining linguistic terms in LTS except those that appear in PLTS, the individual similarity degree and the group similarity degree can make the normalized PLTS more consistent with the true assessment information, thereby improving the accuracy of the normalization method. However, most of the current articles (Lin and Xu 2018; Liao et al. 2017; Zhang et al. 2018b) all adopt the normalization method introduced in Pang et al. (2016), and these pieces of literature are not separately normalizing the incomplete PLTSs according to the different types of the uncertain PL-MCDM problems. Therefore, it is necessary to study the different normalization methods for incomplete PLTSs in the different types of the uncertain PL-MCDM problems.

The projection proposed by Xu (2004) is calculated based on the product of the module sizes of any two vectors having the same dimension and the angle cosine between these two vectors. Since the projection takes into account the modules and the directions of two vectors, the similarity degree between two vectors can be fully reflected, and the calculation of the projection is easy to perform on the computer. Therefore, Zhang et al. (2018a) proposed to apply the projection to solve the uncertain PL-MCDM problems. However, the above projection-based method is used to determine the criteria weights, and the projection has not been applied to normalize the incomplete PLTS in the uncertain PL-MCDM problems. Thus, this paper proposes the projection-based models to normalize the incomplete PLTS. Furthermore, the main idea of the classical two-step method (Xu 2002) is to local optimization firstly, and then recombination weighting. So it is a very effective method for determining the criteria weights values in the MCDM problems where the criteria weights values that are not precisely known but the ranges are available, but in the probabilistic linguistic environment, there is still a gap in the application of this method.

Based on the above analyses, the contributions of this paper are highlighted as follows:

(1) To properly apply the projection to different types of the uncertain PL-MCDM problems, we present the equivalent expression forms of the PLTSs on LTSs $\widetilde{L}$ and $\widehat{L}$, and then the equivalent transformation functions between the PLTS and its associated vector are proposed to perform the conversion between the operations on the PLTSs and the operations on their associated vectors.

(2) After introducing the projection formulas of the PLTSs on LTSs $\widetilde{L}$ and $\widehat{L}$, different normalization models are proposed for different types of the uncertain PL-MCDM problems. And a new deviation degree formula is given to illustrate the projection-based normalization models from the view of theory, which can effectively normalize the incomplete PLTSs in these uncertain PL-MCDM problems.

(3) We propose the probabilistic linguistic two-step method to solve the uncertain PL-MCDM problems where the criteria weights values are not precisely known but the ranges are available, and then the alternatives are sorted via using this decision-making

method. This can enrich the weight-determining method for the uncertain PL-MCDM problems, in which the criteria weights are given in the form of intervals.

The rest of this paper is organized as follows: In Sect. 2, we look back to some basic concepts that need to be used. In Sect. 3, we introduce some new definitions about the PLTSs, such as the equivalent expression forms of the PLTSs, the equivalent transformation functions between the PLTS and its associated vector, the projection formulas of the PLTSs, and a new deviation degree formula. With the projection formulas of the PLTSs, four projection-based normalization models and the probabilistic linguistic two-step method are proposed to solve the uncertain PL-MCDM problems in Sect. 4. A case study about the performance assessment of the data hiding techniques is carried out to illustrate the rationality and validity of our proposed approaches in Sect. 5. In Sect. 6, some comparative analyses are made on the normalization models and the decision-making method. The paper finishes in Sect. 7 with some conclusions and the directions for future studies.

## 2 Preliminaries

In this section, we first introduce the classical two-step method and then review some basic concepts related to the LTSs and the PLTSs that need to be used.

### 2.1 The classical two-step method

The classical two-step method first proposed by Xu (2002) is a simple and objective method for determining criteria weights values and ordering the alternatives. Suppose that the MCDM problem contains a set of alternatives $\varPsi = \{\psi_1, \psi_2, \ldots, \psi_n\}$, a set of criteria $\Theta = \{\theta_1, \theta_2, \ldots, \theta_m\}$ with the weight vector $\boldsymbol{\Omega} = (\omega_1, \omega_2, \ldots, \omega_m)^\top$, where $\omega_i \in [\alpha_i, \beta_i] \subseteq [0, 1]$, $\sum_{i=1}^{m} \omega_i = 1$, $\alpha_i$ and $\beta_i$ are the lower and upper bounds of $\omega_i$, respectively. Then the assessment value $a_{ij}(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ of the alternative $\psi_j$ with respect to a criterion $\theta_i$ can be obtained, and the comprehensive criterion value of the alternative $\psi_j$ is $z_j = \sum_{i=1}^{m} \omega_i a_{ij}(j = 1, 2, \ldots, n)$. The large the comprehensive criterion value $z_j$ is, the optimal the alternative $\psi_j$ will be. Finally, the decision matrix which consists of all assessment values provided by DMs can be constructed as $\boldsymbol{A} = (a_{ij})_{m\times n}$. The classical two-step method consists of two main steps: calculating the optimal criterion weight vector of each alternative and determining the normalized combined weight vector. In summary, the algorithm of the classical two-step method is presented as follows:

**Algorithm 1: (The classical two-step method)**

**Step 1.** Normalize the decision matrix $\boldsymbol{A} = \boldsymbol{(a_{ij})_{m\times n}}$, and the normalized decision matrix denoted as $\boldsymbol{R} = \boldsymbol{(r_{ij})_{m\times n}}$.

**Step 2.** Determine the optimal criterion weight vector $\varOmega_j = \left(\boldsymbol{\omega_1^j}, \omega_2^j, \ldots, \omega_m^j\right)^\top (j = 1, 2, \ldots, n)$ of each alternative via solving the linear model (LP1):

$$\text{(LP1)}\ \ \max z_j = \sum_{i=1}^{m} \omega_i^j r_{ij}$$

$$\text{s.t.} \begin{cases} \omega_i^j \in [\alpha_i, \beta_i] \subset [0, 1], \\ \sum\limits_{i=1}^{m} \omega_i^j = 1. \end{cases}$$

Then the weight matrix $\boldsymbol{W} = (\boldsymbol{\Omega_1}, \boldsymbol{\Omega_2}, \ldots, \boldsymbol{\Omega_n})$ can be obtained.

**Step 3.** Calculate the eigenvector $\boldsymbol{\eta}$ corresponding to the maximum eigenvalue of the matrix $(\boldsymbol{W}^\top \boldsymbol{R})(\boldsymbol{W}^\top \boldsymbol{R})^\top$, then the normalized combined weight vector is $\boldsymbol{\Omega^*} = \boldsymbol{W}\boldsymbol{\eta} = (\omega_1^*, \omega_2^*, \ldots, \omega_m^*)^\top$, where $\sum_{i=1}^m \omega_i^* = 1$.

**Step 4.** Calculate the comprehensive criterion value $z_j^*$ of the alternative $\psi_j$ via formula $z_j^* = \sum_{i=1}^m \omega_i^* r_{ij}$ $(j = 1, 2, \ldots, n)$.

**Step 5.** Use the comprehensive criterion value to rank the alternatives.

**Step 6.** End.

### 2.2 The linguistic term sets

In the actual decision-making processes, DMs are likely to give their assessment information via using linguistic terms, such as “good”, “very good” or “bad”, since the complexity and uncertainty of the objective things and the fuzziness of human thinking. As the basis of linguistic decision-making, the linguistic term set (LTS) has achieved many studies. Up to now, there are two additive LTSs are used widely. The first one is $\widetilde{L} = \{l_0, l_1, \ldots, l_\delta\}$, where the subscripts are all non-negative integers. The second most popular LTS is $\widehat{L} = \{l_t \mid t = -\delta, \ldots, -1, 0, 1, \ldots, \delta\}$, which is a subscript-symmetric additive LTS. Meanwhile, some other LTSs have been developed as well. For instance, unbalanced additive LTS (Dai et al. 2008), unbalanced multiplicative LTS (Xu 2010), and so on. Different LTSs have different application environments, and in general, each LTS contains the linguistic term “indifference”. For example, for evenly distributed linguistic terms which are used for assessment, the LTS $\widetilde{L} = \{l_0, l_1, \ldots, l_\delta\}$ is very suitable for tackling the situation where all the linguistic terms are on the same side of the linguistic term “indifference”, but if all the linguistic terms are on the opposite side of the linguistic term “indifference”, the LTS $\widehat{L} = \{l_t \mid t = -\delta, \ldots, -1, 0, 1, \ldots, \delta\}$ will be more appropriate. In the following, the research content of this paper is mainly based on these two LTSs $\widetilde{L}$ and $\widehat{L}$.

### 2.3 The probabilistic linguistic term sets

Concerning the shortcoming of the hesitant fuzzy linguistic term set (HFLTS) (Rodríguez et al. 2012) which cannot express the true probability of each possible linguistic term, Pang et al. (2016) proposed the probabilistic linguistic term set (PLTS), which consists of all possible linguistic terms and their corresponding probabilities.

**Definition 1** (Pang et al. 2016) The probabilistic linguistic term set (PLTS) $PL(p)$ on LTS $\widetilde{L} = \{l_0, l_1, \ldots, l_\delta\}$ can be defined as:

$$PL(p) = \left\{ l_{(k)}\big(p_{(k)}\big) \,\middle|\, l_{(k)} \in \widetilde{L},\, p_{(k)} \geq 0, k = 1, 2, \ldots, \#PL(p), \sum_{k=1}^{\#PL(p)} p_{(k)} \leq 1 \right\},$$

and the PLTS $PL(p)$ based on the LTS $\widehat{L} = \{l_t \mid t = -\delta, \ldots, -1, 0, 1, \ldots, \delta\}$ can be expressed as:

$$PL(p) = \left\{ l_{(k)}\big(p_{(k)}\big) \,\middle|\, l_{(k)} \in \widehat{L},\, p_{(k)} \geq 0, k = 1, 2, \ldots, \#PL(p), \sum_{k=1}^{\#PL(p)} p_{(k)} \leq 1 \right\}.$$

where the probabilistic linguistic term element (PLTE) $l_{(k)}(p_{(k)})$ consists of the linguistic term $l_{(k)}$ and its associated probability $p_{(k)}$, and $\#PL(p)$ is the cardinality of the PLTS $PL(p)$.

Besides, for the multiple DMs PL-MCDM problems, Wu and Liao (2018) developed an aggregation formula to integrate the assessments provided by multiple DMs to a group assessment.

**Definition 2** (Wu and Liao 2018) Given a set of DMs $D = \{d^{(q)} \mid q = 1, 2, \ldots, Q\}$ whose weight vector is $\big(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(Q)}\big)^\top \Big(\sum_{q=1}^{Q} \gamma^{(q)} = 1\Big)$ and the LTS is $L$ ($L$ may be $\widetilde{L}$, $\widehat{L}$ or some other LTS). Suppose that the assessment of the alternative $\psi_j$ with respect to a criterion $\theta_i$ given by DM $d^{(q)}$ is

$$\begin{aligned} PL_{ij}^{(q)}(p) = \Bigg\{ & l_{(k)}^{(q)}\big(p_{(k)}^{(q)}\big) \,\Big|\, l_{(k)}^{(q)} \in L,\, p_{(k)}^{(q)} \\ & \geq 0, k = 1, 2, \ldots, \#PL_{ij}^{(q)}(p), \sum_{k=1}^{\#PL_{ij}^{(q)}(p)} p_{(k)}^{(q)} \leq 1 \Bigg\}, \end{aligned}$$

and the number of all different linguistic terms in $PL_{ij}^{(1)}(p), PL_{ij}^{(2)}(p), \ldots, PL_{ij}^{(Q)}(p)$ is $K$. Then the group assessment expressed in PLTS is

$$PL_{ij}(p) = \left\{ l_{(k)}\big(p_{(k)}\big) \,\middle|\, l_{(k)} \in L, k = 1, 2, \ldots, K,\, p_{(k)} = \sum_{q=1}^{Q} \xi_{(k)}^{(q)} \gamma^{(q)} \right\},$$

where $\xi_{(k)}^{(q)}$ is the probability of the linguistic term $l_{(k)}^{(q)}$ in $PL_{ij}^{(q)}(p)$ and

$$\xi_{(k)}^{(q)} = \begin{cases} p_{(k)}^{(q)} & \text{if } l_{(k)}^{(q)} \in PL_{ij}^{(q)}(p), \\ 0 & \text{otherwise.} \end{cases}$$

## 3 Some new definitions of the PLTSs

In this section, we firstly introduce the equivalent expression forms of the PLTSs on the LTSs $\widetilde{L}$ and $\widehat{L}$. Then, based on these equivalent expression forms, the equivalent transformation functions between the PLTS and its associated vector are proposed to realize the conversion between the operations on the PLTSs and the operations on their associated vectors. Finally, the projection formulas of the PLTSs on the LTSs $\widetilde{L}$ and $\widehat{L}$ are developed and to accurately measure the deviation degree between the PLTSs, a new deviation degree formula for the PLTSs is developed.

### 3.1 The equivalent expression forms of the PLTSs

For the PLTS given above, if each PLTE in the PLTS is considered to be one dimension, then different PLTSs may have different dimensions. However, the premise of the projection operation is two vectors having the same dimension, so before converting PLTSs having different dimensions into vectors having the same dimension, the PLTSs having different dimensions must be equivalently converted into PLTSs having the same dimension. To do this, we present the equivalent expression forms of the PLTSs on the LTSs $\widetilde{L}$ and $\widehat{L}$ as follows.

**Definition 3** Let $PL(p)$ be a PLTS on LTS $\widetilde{L} = \{l_0, l_1, \ldots, l_\delta\}$, then its equivalent expression form is defined as:

$$\widetilde{PL}(p) = \left\{ l_{(k-1)}\left(\widetilde{p}_{(k)}\right) \,\middle|\, \widetilde{p}_{(k)} \geq 0, k = 1, 2, \ldots, \delta + 1, \sum_{k=1}^{\delta+1} \widetilde{p}_{(k)} \leq 1 \right\},$$

where $\widetilde{p}_{(k)} = p_{(k)}$ if $l_{(k-1)}(p_{(k)}) \in PL(p)$, otherwise, $\widetilde{p}_{(k)} = 0$.

And for the PLTS based on the LTS $\widehat{L}$, we develop its equivalent expression form as follows.

**Definition 4** Let $PL(p)$ be a PLTS on LTS $\widehat{L} = \{l_t \mid t = -\delta, \ldots, -1, 0, 1, \ldots, \delta\}$, then its equivalent expression form can be defined as:

$$\widehat{PL}(p) = \left\{ l_{(k-\delta-1)}\left(\widehat{p}_{(k)}\right) \,\middle|\, \widehat{p}_{(k)} \geq 0, k = 1, 2, \ldots, 2\delta + 1, \sum_{k=1}^{2\delta+1} \widehat{p}_{(k)} \leq 1 \right\},$$

where $\widehat{p}_{(k)} = p_{(k)}$ if $l_{(k-\delta-1)}(p_{(k)}) \in PL(p)$, otherwise, $\widehat{p}_{(k)} = 0$.

*Remark 1* Compared by the expressions of the PLTSs in Definition 1, we can see that the above two equivalent expression forms of the PLTSs have the following advantages:

(1) For any two PLTSs on the same LTS, their equivalent expression forms will have the same linguistic terms and the probabilities of all linguistic terms in the original PLTSs are consistent before and after the change of expression forms. Therefore, the equivalent representation of PLTSs did not change the initial evaluation information given by DMs.

(2) The equivalent expression forms of PLTSs can eliminate the influence of different dimensions, thereby avoiding the difficulties caused by different dimensions in the future.

### 3.2 The equivalent transformation functions of the PLTSs

Based on the equivalent expression forms of the PLTSs on LTSs $\widetilde{L}$ and $\widehat{L}$, the equivalent transformation functions developed below can map the PLTSs into a high-dimensional space and obtain their associated vectors with the same dimension, so that the conversion can be implemented between the PLTS and its associated vector $\boldsymbol{v}$.

**Definition 5** Suppose $\widetilde{PL}(p) = \left\{ l_{(k-1)}\left(\widetilde{p}_{(k)}\right) \mid \widetilde{p}_{(k)} \geq 0, k = 1, 2, \ldots, \delta + 1, \sum_{k=1}^{\delta+1} \widetilde{p}_{(k)} \leq 1 \right\}$ is the equivalent expression form of the PLTS $PL(p)$ on LTS $\widetilde{L} = \{l_0, l_1, \ldots, l_\delta\}$, and its associated vector is $\boldsymbol{v} = \left(v_1, v_2, \ldots, v_{\delta+1}\right)$. Then $l_{(k-1)}\left(\widetilde{p}_{(k)}\right)$ and $v_k$ can be transformed into each other by the functions $\sigma$ and $\sigma^{-1}$ given as:

$$
\begin{aligned}
\sigma \ &: \widetilde{PL}(p) \to \boldsymbol{v} \\
&\quad l_{(k-1)}\left(\widetilde{p}_{(k)}\right) \mapsto \sigma\left(l_{(k-1)}\left(\widetilde{p}_{(k)}\right)\right) = \frac{k}{\delta+1}\widetilde{p}_{(k)} = v_k, \\
\sigma^{-1} \ &: \boldsymbol{v} \to \widetilde{PL}(p) \\
&\quad v_k \mapsto \sigma^{-1}(v_k) = l_{(k-1)}\left(\frac{\delta+1}{k}v_k\right).
\end{aligned}
$$

Thus, for a PLTS $PL(p)$ on LTS $\widetilde{L}$, its associated vector $\boldsymbol{v} = \left(\frac{1}{\delta+1}\widetilde{p}_{(1)}, \frac{2}{\delta+1}\widetilde{p}_{(2)}, \ldots, \frac{\delta}{\delta+1}\widetilde{p}_{(\delta)}, \widetilde{p}_{(\delta+1)}\right)$ can be obtained via Definitions 3 and 5.

Similarly, the equivalent transformation functions between the PLTS on LTS $\widehat{L}$ and its associated vector $\boldsymbol{v}$ are given below.

**Definition 6** Suppose $\widehat{PL}(p) = \left\{ l_{(k-\delta-1)}\left(\widehat{p}_{(k)}\right) \mid \widehat{p}_{(k)} \geq 0, k = 1, 2, \ldots, 2\delta + 1, \sum_{k=1}^{2\delta+1} \widehat{p}_{(k)} \leq 1 \right\}$ is the equivalent expression form of the PLTS $PL(p)$ on LTS $\widehat{L} = \left\{ l_t \mid t = -\delta, \ldots, -1, 0, 1, \ldots, \delta \right\}$, and $\boldsymbol{v} = \left(v_1, v_2, \ldots, v_{2\delta+1}\right)$ is the associated vector of the PLTS $PL(p)$. Then $l_{(k-\delta-1)}\left(p_{(k)}\right)$ and $v_k$ can be transformed into each other by the functions $\rho$ and $\rho^{-1}$ given as:

$$
\begin{aligned}
\rho \ &: \widehat{PL}(p) \to \boldsymbol{v} \\
&\quad l_{(k-\delta-1)}\left(\widehat{p}_{(k)}\right) \mapsto \rho\left(l_{(k-\delta-1)}\left(\widehat{p}_{(k)}\right)\right) = \frac{2k-2\delta-3}{4\delta+2}\widehat{p}_{(k)} = v_k, \\
\rho^{-1} \ &: \boldsymbol{v} \to \widehat{PL}(p) \\
&\quad v_k \mapsto \rho^{-1}(v_k) = l_{(k-\delta-1)}\left(\frac{4\delta+2}{2k-2\delta-3}v_k\right).
\end{aligned}
$$

Therefore, the associated vector $\boldsymbol{v} = \left( \frac{-1-2\delta}{4\delta+2} \widehat{p}_{(1)}, \frac{1-2\delta}{4\delta+2} \widehat{p}_{(2)}, \ldots, \frac{2\delta-3}{4\delta+2} \widehat{p}_{(2\delta)}, \frac{2\delta-1}{4\delta+2} \widehat{p}_{(2\delta+1)} \right)$ of the PLTS $PL(p)$ on LTS $\widehat{L}$ can be obtained according to Definitions 4 and 6.

After that, the operations on the PLTSs based on LTS $\widetilde{L}$ or $\widehat{L}$ and the operations on their associated vectors can be converted by these equivalent transformation functions.

### 3.3 The projection formulas of the PLTSs

Since the projection formula is given based on any two vectors with the modules and the angle between them, and the PLTS is given in the form of a set, the PLTSs must be converted to their associated vectors according to the Definitions 3–6 before giving their projection formulas. To express simply these projection formulas, we use the projection formulas of their associated vectors to represent the projection formulas of these two PLTSs in this paper. Thus the projection formulas of the PLTSs on LTS $\widetilde{L} = \{l_0, l_1, \ldots, l_\delta\}$ can be defined as follows.

**Definition 7** Suppose $\widetilde{PL}_1(p)$ and $\widetilde{PL}_2(p)$ are the equivalent expression forms of the PLTSs $PL_1(p)$ and $PL_2(p)$ on LTS $\widetilde{L}$, $\boldsymbol{v_1} = \left( \frac{1}{\delta+1} \widetilde{p}_{1(1)}, \frac{2}{\delta+1} \widetilde{p}_{1(2)}, \ldots, \frac{\delta}{\delta+1} \widetilde{p}_{1(\delta)}, \widetilde{p}_{1(\delta+1)} \right)$ and $\boldsymbol{v_2} = \left( \frac{1}{\delta+1} \widetilde{p}_{2(1)}, \frac{2}{\delta+1} \widetilde{p}_{2(2)}, \ldots, \frac{\delta}{\delta+1} \widetilde{p}_{2(\delta)}, \widetilde{p}_{2(\delta+1)} \right)$ are their associated vectors, respectively. Then the cosine formula related to the PLTSs $PL_1(p)$ and $PL_2(p)$ is defined as follows:

$$\cos\theta = \cos\langle \boldsymbol{v_1}, \boldsymbol{v_2} \rangle = \frac{\boldsymbol{v_1} \boldsymbol{v_2}}{|\boldsymbol{v_1}||\boldsymbol{v_2}|} = \frac{\sum_{k=1}^{\delta+1} \left( \frac{k}{\delta+1} \right)^2 \widetilde{p}_{1(k)} \widetilde{p}_{2(k)}}{\sqrt{\sum_{k=1}^{\delta+1} \left( \frac{k}{\delta+1} \widetilde{p}_{1(k)} \right)^2} \times \sqrt{\sum_{k=1}^{\delta+1} \left( \frac{k}{\delta+1} \widetilde{p}_{2(k)} \right)^2}},$$

where $|\boldsymbol{v_1}|$ and $|\boldsymbol{v_2}|$ are the modules of vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ respectively, and $\boldsymbol{v_1} \boldsymbol{v_2}$ is the inner product of two vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

Therefore, the projection formula of the PLTS $PL_1(p)$ on the PLTS $PL_2(p)$ (see Fig.1) is:

$$\begin{aligned} \mathrm{Prj}_{\boldsymbol{v_2}}(\boldsymbol{v_1}) &= |\boldsymbol{v_1}| \cos\theta = \frac{\boldsymbol{v_1} \boldsymbol{v_2}}{|\boldsymbol{v_2}|} \\ &= \left( \sum_{k=1}^{\delta+1} \left( \frac{k}{\delta+1} \right)^2 \widetilde{p}_{1(k)} \widetilde{p}_{2(k)} \right) \Bigg/ \sqrt{\sum_{k=1}^{\delta+1} \left( \frac{k}{\delta+1} \widetilde{p}_{2(k)} \right)^2}, \end{aligned}$$

and the projection formula of the PLTS $PL_2(p)$ on the PLTS $PL_1(p)$ (see Fig.2) is:

$$\mathrm{Prj}_{\boldsymbol{v_1}}(\boldsymbol{v_2}) = |\boldsymbol{v_2}| \cos\theta = \frac{\boldsymbol{v_1} \boldsymbol{v_2}}{|\boldsymbol{v_1}|} = \left( \sum_{k=1}^{\delta+1} \left( \frac{k}{\delta+1} \right)^2 \widetilde{p}_{1(k)} \widetilde{p}_{2(k)} \right) \Bigg/ \sqrt{\sum_{k=1}^{\delta+1} \left( \frac{k}{\delta+1} \widetilde{p}_{1(k)} \right)^2}.$$

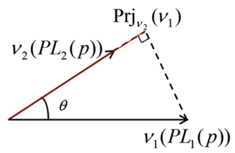


**Fig. 1** The projection of the PLTS $PL_1(p)$ on $PL_2(p)$

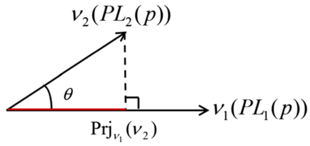


**Fig. 2** The projection of the PLTS $PL_2(p)$ on $PL_1(p)$

Obviously, the greater the values of $\mathrm{Prj}_{\boldsymbol{v_2}}(\boldsymbol{v_1})$ and $\mathrm{Prj}_{\boldsymbol{v_1}}(\boldsymbol{v_2})$, the smaller the deviation degree between the PLTSs $PL_1(p)$ and $PL_2(p)$ on LTS $\widetilde{L}$.

Similarly, the projection formulas of the PLTSs on LTS $\widehat{L} = \{l_t \mid t = -\delta, \ldots, -1, 0, 1, \ldots, \delta\}$ are given as follows.

**Definition 8** Suppose $\widehat{PL}_1(p)$ and $\widehat{PL}_2(p)$ are the equivalent expression forms of the PLTSs $PL_1(p)$ and $PL_2(p)$ on LTS $\widehat{L}$, $\boldsymbol{v_1} = \left(\frac{-1-2\delta}{4\delta+2}\widehat{p}_{1(1)}, \frac{1-2\delta}{4\delta+2}\widehat{p}_{1(2)}, \ldots, \frac{2\delta-3}{4\delta+2}\widehat{p}_{1(2\delta)}, \frac{2\delta-1}{4\delta+2}\widehat{p}_{1(2\delta+1)}\right)$ and $\boldsymbol{v_2} = \left(\frac{-1-2\delta}{4\delta+2}\widehat{p}_{2(1)}, \frac{1-2\delta}{4\delta+2}\widehat{p}_{2(2)}, \ldots, \frac{2\delta-3}{4\delta+2}\widehat{p}_{2(2\delta)}, \frac{2\delta-1}{4\delta+2}\widehat{p}_{2(2\delta+1)}\right)$ are their associated vectors, respectively. Then the cosine formula related to the PLTSs $PL_1(p)$ and $PL_2(p)$ is defined as:

$$\cos\theta = \cos\langle \boldsymbol{v_1}, \boldsymbol{v_2}\rangle = \frac{\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\right)^2 \widehat{p}_{1(k)}\widehat{p}_{2(k)}}{\sqrt{\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\widehat{p}_{1(k)}\right)^2} \times \sqrt{\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\widehat{p}_{2(k)}\right)^2}}.$$

Therefore, the projection formula of the PLTS $PL_1(p)$ on the PLTS $PL_2(p)$ is:

$$\begin{aligned}\mathrm{Prj}_{\boldsymbol{v_2}}(\boldsymbol{v_1}) &= |\boldsymbol{v_1}|\cos\theta \\ &= \frac{\boldsymbol{v_1}\boldsymbol{v_2}}{|\boldsymbol{v_2}|} = \left(\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\right)^2 \widehat{p}_{1(k)}\widehat{p}_{2(k)}\right) \Big/ \sqrt{\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\widehat{p}_{2(k)}\right)^2},\end{aligned}$$

and the projection formula of the PLTS $PL_2(p)$ on the PLTS $PL_1(p)$ is:

$$\begin{aligned}\mathrm{Prj}_{\boldsymbol{v_1}}(\boldsymbol{v_2}) &= |\boldsymbol{v_2}|\cos\theta \\ &= \frac{\boldsymbol{v_1}\boldsymbol{v_2}}{|\boldsymbol{v_1}|} = \left(\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\right)^2 \widehat{p}_{1(k)}\widehat{p}_{2(k)}\right) \Big/ \sqrt{\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\widehat{p}_{1(k)}\right)^2}.\end{aligned}$$

Obviously, the greater the values of $\mathrm{Prj}_{\boldsymbol{v_2}}(\boldsymbol{v_1})$ and $\mathrm{Prj}_{\boldsymbol{v_1}}(\boldsymbol{v_2})$, the smaller the deviation degree between the PLTSs $PL_1(p)$ and $PL_2(p)$ on LTS $\widehat{L}$.

To accurately measure the deviation degree between two PLTSs, in what follows, we develop a new deviation degree formula for the PLTSs.

### 3.4 The new deviation degree formula for the PLTSs

For two PLTSs $PL_1(p)$ and $PL_2(p)$ with exactly the same linguistic terms, and the probabilities of each corresponding two linguistic terms in $PL_1(p)$ and $PL_2(p)$ are not all equal to 0, below a new deviation degree formula is given to reflect the deviation degree between the these two PLTSs.

**Definition 9** Suppose PLTSs $PL_1(p) = \{l_{(k)}(p_{1(k)}) \mid p_{1(k)} \geq 0, k = 1, 2, \ldots, K', \sum_{k=1}^{K'} p_{1(k)} \leq 1\}$ and $PL_2(p) = \{l_{(k)}(p_{2(k)}) \mid p_{2(k)} \geq 0, k = 1, 2, \ldots, K', \sum_{k=1}^{K'} p_{2(k)} \leq 1\}$ have the same linguistic terms on the same LTS, and the probabilities of each corresponding two linguistic terms in $PL_1(p)$ and $PL_2(p)$ are not all equal to 0. Then the deviation degree between the PLTSs $PL_1(p)$ and $PL_2(p)$ is defined as:

$$d\big(PL_1(p), PL_2(p)\big) = \left(\sum_{k=1}^{K'} \frac{1}{K'}\Big(\varphi_{(k)}\big(p_{1(k)} - p_{2(k)}\big)\Big)^2\right)^{1/2},$$

where $K'$ is the number of all the different linguistic terms in the PLTSs $PL_1(p)$ and $PL_2(p)$, $\varphi_{(k)}$ is the subscript of the linguistic term $l_{(k)}$, $p_{1(k)}$ and $p_{2(k)}$ are the associated probabilities of the linguistic term $l_{(k)}$ in $PL_1(p)$ and $PL_2(p)$, respectively.

However, for any two PLTSs $PL_1(p)$ and $PL_2(p)$ on the same LTS, it is normal that these two PLTSs have different linguistic terms. For instance, let $PL_1(p) = \{l_3(0.3), l_4(0.2), l_5(0.5)\}$ and $PL_2(p) = \{l_2(0.4), l_3(0.6)\}$ be two PLTSs. Obviously, the linguistic terms $l_4$ and $l_5$ in $PL_1(p)$ do not appear in $PL_2(p)$, and the linguistic term $l_2$ in $PL_2(p)$ is not reflected in $PL_1(p)$ as well. Therefore, the linguistic terms in PLTSs $PL_1(p)$ and $PL_2(p)$ are not the same. In this case, we have to perform the following two steps before calculating the deviation degree between these two PLTSs.

(a) Construct a new LTS consisting of all the different linguistic terms that appear in the PLTS $PL_1(p)$ or the PLTS $PL_2(p)$.

(b) Add the PLTEs so that the numbers of the PLTEs in these two PLTSs are the same. For a linguistic term in $PL_1(p)$ or $PL_2(p)$ that do not appear in the new LTS, we should add it to $PL_1(p)$ or $PL_2(p)$ until all linguistic terms in the new LTS can be found in the PLTSs $PL_1(p)$ and $PL_2(p)$, and the probability of the added linguistic term is zero.

Next, calculate the deviation degree between two added PLTSs with the same linguistic terms based on the new LTS via Definition 9. Below we use an example to demonstrate the calculation processes of this new deviation degree formula.

*Example 1* Suppose $PL_1(p) = \{l_3(0.25), l_5(0.1)\}$, $PL_2(p) = \{l_3(0.3), l_4(0.1), l_5(0.5), l_6(0.1)\}$ and $PL_3(p) = \{l_3(0.71), l_5(0.29)\}$ are three PLTSs, then we calculate the deviation degrees $d\big(PL_1(p), PL_2(p)\big)$ and $d\big(PL_1(p), PL_3(p)\big)$.

(1) It is obviously that the linguistic terms in PLTSs $PL_1(p)$ and $PL_2(p)$ are not exactly the same, so for the PLTSs $PL_1(p)$ and $PL_2(p)$, we need to determine a new LTS $L = \{l_3, l_4, l_5, l_6\}$ in the first place. Then, the added PLTS $PL_1(p)$ on the new LTS $L = \left\{l_3, l_4, l_5, l_6\right\}$ is $PL_1'(p) = \left\{l_3(0.25), l_4(0), l_5(0.1), l_6(0)\right\}$ and the added PLTS $PL_2(p)$ on the new LTS $L = \left\{l_3, l_4, l_5, l_6\right\}$ is $PL_2'(p) = PL_2(p) = \left\{l_3(0.3), l_4(0.1), l_5(0.5), l_6(0.1)\right\}$. Finally, the deviation degree $d\big(PL_1(p), PL_2(p)\big) = d\big(PL_1'(p), PL_2'(p)\big) = \Big(\frac{1}{4}\big((3 \times (0.25 - 0.3))^2 + (4 \times (0 - 0.1))^2 + (5 \times (0.1 - 0.5))^2 + (6 \times (0 - 0.1))^2\big)\Big)^{1/2} = 1.066$.

(2) For the PLTSs $PL_1(p)$ and $PL_3(p)$ with the identical linguistic terms, then according to Definition 9, the deviation degree between the PLTSs $PL_1(p)$ and $PL_3(p)$ is $d\big(PL_1(p), PL_3(p)\big) = \Big(\frac{1}{2}\big((3 \times (0.25 - 0.71))^2 + (5 \times (0.1 - 0.29))^2\big)\Big)^{1/2} = 1.185$.

*Remark 2* This new deviation degree formula can more accurately reflect the deviation degree between two PLTSs than the deviation degree proposed in Pang et al. (2016). For example, let $PL_1(p) = \{l_1(0.3), l_2(0.3)\}$ and $PL_2(p) = \{l_2(0.3), l_3(0.1)\}$ be two PLTSs, then the deviation degree between the PLTSs $PL_1(p)$ and $PL_2(p)$ via the deviation degree formula proposed in this paper is $d\big(PL_1(p), PL_2(p)\big) = 0.245$, while according to the deviation degree introduced in Pang et al. (2016), we obtain $d'\big(PL_1(p), PL_2(p)\big) = 0$. Obviously, the accuracy of this new deviation degree formula is relatively high.

From Definition 9, we can find that if the deviation degree between two PLTSs is smaller, the value of their corresponding similarity degree will be larger. Therefore, we could use this deviation degree formula to calculate the similarity degrees between the PLTSs and their normalization forms to show the feasibility and efficiency of the projection-based normalization models proposed in this paper from the view of theory.

## 4 The solutions of the PL-MCDM problems based on double information under imperfect conditions

Since the uncertain single DM PL-MCDM problem is a special case of the uncertain multiple DMs PL-MCDM problem, in this paper, we mainly study the projection-based normalization models and the probabilistic linguistic two-step method to tackle the uncertain multiple DMs PL-MCDM problems, in which multiple DMs are invited to participate in the evaluation of these problems. And for the convenience of presentation, we only consider the PLTS on LTS $\widetilde{L}$ and all PLTEs in the PLTS $PL(p)$ are arranged according to the subscripts of the linguistic terms in ascending order. First, we make the following provisions for the symbols that appear in the uncertain multiple DMs PL-MCDM problem.

### 4.1 Description and symbols clarification of the uncertain multiple DMs PL-MCDM problem

Suppose that a uncertain multiple DMs PL-MCDM problem contains a set of DMs $D = \{d^{(q)} \mid q = 1, 2, \ldots, Q\}$ whose weight vector is $(\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(Q)})^{\top}(\sum_{q=1}^{Q} \gamma^{(q)} = 1)$, a set of alternatives $\Psi = \{\psi_1, \psi_2, \ldots, \psi_n\}$ and a set of criteria $\Theta = \{\theta_1, \theta_2, \ldots, \theta_m\}$ with the weight vector $\mathbf{\Omega} = \left(\omega_1, \omega_2, \ldots, \omega_m\right)^{\top}$, where $\omega_i \in [\alpha_i, \beta_i] \subseteq [0, 1], \sum_{i=1}^{m} \omega_i = 1, \alpha_i$ and $\beta_i$ are the lower and upper bounds of $\omega_i$, respectively. And the assessment of the alternative $\psi_j$ with respect to a criterion $\theta_i$ provided by DM $d^{(q)}$ is denoted as a PLTS $PL_{ij}^{(q)}(p) = \{l_{(k)}^{(q)}(p_{(k)}^{(q)}) \mid l_{(k)}^{(q)} \in L, p_{(k)}^{(q)} \geq 0, k = 1, 2, \ldots, \#PL_{ij}^{(q)}(p), \sum_{k=1}^{\#PL_{ij}^{(q)}(p)} p_{(k)}^{(q)} \leq 1\}(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; q = 1, 2, \ldots, Q)$ via using the LTS $\widetilde{L} = \{l_t \mid t = 0, 1, \ldots, \delta\}$. Finally, the probabilistic linguistic decision matrixes which consist of all PLTSs given by each DM can be constructed as $\boldsymbol{PL^{(q)}} = (PL_{ij}^{(q)}(p))_{m\times n}$ $(q = 1, 2, \ldots, Q)$.

### 4.2 The projection-based normalization models of the PLTSs

Corresponding to the equivalent expression forms of the PLTSs, in the following, we proposed the normalization forms of the PLTS $PL_{ij}^{(q)}(p)(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; q = 1, 2, \ldots, Q)$.

**Definition 10** The normalization of the PLTS $PL_{ij}^{(q)}(p)$ on LTS $\widetilde{L} = \{l_0, l_1, \ldots, l_\delta\}$ is defined by:

$$\widetilde{PL}_{ij}^{*(q)}(p) = \left\{l_{(k-1)}\left(p_{ij(k)}^{*(q)}\right) \,\middle|\, p_{ij(k)}^{*(q)} \geq \widetilde{p}_{ij(k)}^{(q)}, k = 1, 2, \ldots, \delta + 1, \sum_{k=1}^{\delta+1} p_{ij(k)}^{*(q)} = 1\right\},$$

and the normalization of $PL_{ij}^{(q)}(p)$ on LTS $\widehat{L} = \left\{l_t \mid t = -\delta, \ldots, -1, 0, 1, \ldots, \delta\right\}$ can defined as:

$$\widehat{PL}_{ij}^{*(q)}(p) = \left\{l_{(k-\delta-1)}\left(p_{ij(k)}^{*(q)}\right) \,\middle|\, p_{ij(k)}^{*(q)} \geq \widehat{p}_{ij(k)}^{(q)}, k = 1, 2, \ldots, 2\delta + 1, \sum_{k=1}^{2\delta+1} p_{ij(k)}^{*(q)} = 1\right\},$$

where $l_{(k-1)}\left(\widetilde{p}_{ij(k)}^{(q)}\right) \in \widetilde{PL}_{ij}^{(q)}(p)$, $l_{(k-\delta-1)}\left(\widehat{p}_{ij(k)}^{(q)}\right) \in \widehat{PL}_{ij}^{(q)}(p)$, $\widetilde{PL}_{ij}^{(q)}(p)$ is the equivalent expression form of the PLTS $PL_{ij}^{(q)}(p)$ on LTS $\widetilde{L}$ and $\widehat{PL}(p)$ is the equivalent expression form of the PLTS $PL_{ij}^{(q)}(p)$ on LTS $\widehat{L}$.

*Remark 3* For the PLTS $\widetilde{PL}^{*}(p)$ which is the normalization of the PLTS $PL(p)$ on LTS $\widetilde{L}$, it can be seen that the probability of each linguistic term in the PLTS $\widetilde{PL}^{*}(p)$ is

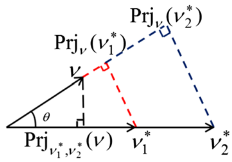


**Fig. 3** The projections of any two vectors of three vectors $\boldsymbol{v}$, $\boldsymbol{v}_1^*$ and $\boldsymbol{v}_2^*$ for the same angle $\theta$

greater than or equal to the probability of the same linguistic term in the PLTS $PL(p)$. Therefore, the modular of the associated vector $\boldsymbol{v}^*$ of the PLTS $PL^*(p)$ must be not less than the modular of the associated vector $\boldsymbol{v}$ of the PLTS $PL(p)$, in other words, $|\boldsymbol{v}| \le |\boldsymbol{v}^*|$. Besides, for the LTS $\widehat{L}$, the above results are still valid.

Therefore, for an alternative $\psi_j$ with respect to a criterion $\theta_i$, the projection formulas of the PLTS $\widetilde{PL}_{ij}^{*(q)}(p)$ and the PLTS $PL_{ij}^{(p)}(p)$ which is provided by DM $d^{(p)}\big(p, q = 1, 2, \ldots, Q\big)$ are defined as follows:

$$\mathrm{Prj}_{\boldsymbol{v}_{ij}^{(p)}}\big(\boldsymbol{v}_{ij}^{*(q)}\big) = \big|\boldsymbol{v}_{ij}^{*(q)}\big| \cos\theta = \frac{\boldsymbol{v}_{ij}^{*(q)}\boldsymbol{v}_{ij}^{(p)}}{\big|\boldsymbol{v}_{ij}^{(p)}\big|} = \left(\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}\right)^2 p_{ij(k)}^{*(q)}\widetilde{p}_{ij(k)}^{(p)}\right) \Big/ \sqrt{\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}\widetilde{p}_{ij(k)}^{(p)}\right)^2}, \ \mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\big(\boldsymbol{v}_{ij}^{(p)}\big) = \big|\boldsymbol{v}_{ij}^{(p)}\big| \cos\theta$$

$$= \frac{\boldsymbol{v}_{ij}^{*(q)}\boldsymbol{v}_{ij}^{(p)}}{\big|\boldsymbol{v}_{ij}^{*(q)}\big|} = \left(\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}\right)^2 p_{ij(k)}^{*(q)}\widetilde{p}_{ij(k)}^{(p)}\right) \Big/ \sqrt{\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}p_{ij(k)}^{*(p)}\right)^2},$$

where $\boldsymbol{v}_{ij}^{(p)} = \left(\frac{1}{\delta+1}\widetilde{p}_{ij(1)}^{(p)}, \frac{2}{\delta+1}\widetilde{p}_{ij(2)}^{(p)}, \ldots, \frac{\delta}{\delta+1}\widetilde{p}_{ij(\delta)}^{(p)}, \widetilde{p}_{ij(\delta+1)}^{(p)}\right)$ and $\boldsymbol{v}_{ij}^{*(q)} = \left(\frac{1}{\delta+1}p_{ij(1)}^{*(q)}, \frac{2}{\delta+1}p_{ij(2)}^{*(q)}, \cdots, \frac{\delta}{\delta+1}p_{ij(\delta)}^{*(q)}, p_{ij(\delta+1)}^{*(q)}\right)$ are the associated vectors of the PLTSs $\widetilde{PL}_{ij}^{(p)}(p)$ and $\widetilde{PL}_{ij}^{*(q)}(p)$.

Obviously, the greater the values of $\mathrm{Prj}_{\boldsymbol{v}_{ij}^{(p)}}\big(\boldsymbol{v}_{ij}^{*(q)}\big)$ and $\mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\big(\boldsymbol{v}_{ij}^{(p)}\big)$, the smaller the deviation degree between the PLTSs $PL_{ij}^{(p)}(p)$ and $\widetilde{PL}_{ij}^{*(q)}(p)$, that is, the better the normalized PLTS $\widetilde{PL}_{ij}^{*(q)}(p)$ is.

However, from Figs. 1 and 2, we find that the projection of the PLTS $\widetilde{PL}_{ij}^{*(q)}(p)$ on the PLTS $PL_{ij}^{(p)}(p)$ is generally unequal to the projection of the PLTS $PL_{ij}^{(p)}(p)$ on the PLTS $\widetilde{PL}_{ij}^{*(q)}(p)$. Therefore, when we construct the normalization model to reduce the deviation degree between the PLTS $PL_{ij}^{(q)}(p)$ and its normalization form $\widetilde{PL}_{ij}^{*(q)}(p)$, both of these projection formulas need to be taken into account.

Furthermore, for three vectors $|\boldsymbol{v}| < |\boldsymbol{v}_1^*| < |\boldsymbol{v}_2^*|$ and the same angle $\theta$ (see Fig. 3), the value of $\mathrm{Prj}_{\boldsymbol{v}}(\boldsymbol{v}^*)$ increases as the modular of vector $\boldsymbol{v}^*$ increases, but the projection of the PLTS $PL(p)$ on the PLTS $\widetilde{PL}^*(p)$ does not change. At this point, $\mathrm{Prj}_{\boldsymbol{v}^*}(\boldsymbol{v})$ no longer plays an effective role, and we could replace the $\mathrm{Prj}_{\boldsymbol{v}^*}(\boldsymbol{v})$ with $\frac{1}{|\boldsymbol{v}^*|}\mathrm{Prj}_{\boldsymbol{v}^*}(\boldsymbol{v})$, where $0 \le \frac{1}{|\boldsymbol{v}^*|}\mathrm{Prj}_{\boldsymbol{v}^*}(\boldsymbol{v}) \le 1$. And the greater the value of $\frac{1}{|\boldsymbol{v}^*|}\mathrm{Prj}_{\boldsymbol{v}^*}(\boldsymbol{v})$, the smaller the deviation degree between the PLTS $PL(p)$ and the PLTS $\widetilde{PL}^*(p)$, and thus the better the normalized PLTS $\widetilde{PL}^*(p)$. Similarly, for the projection formula of the PLTS $PL_{ij}^{(p)}(p)$ on the PLTS $\widetilde{PL}_{ij}^{*(q)}(p)$, we also use $\frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\left(\boldsymbol{v}_{ij}^{(p)}\right)$ instead of $\mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\left(\boldsymbol{v}_{ij}^{(p)}\right)$, and the formula of $\frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\left(\boldsymbol{v}_{ij}^{(p)}\right)$ is given as follows:

$$\frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\left(\boldsymbol{v}_{ij}^{(p)}\right) = \frac{\left|\boldsymbol{v}_{ij}^{(p)}\right|}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\cos\theta = \frac{\boldsymbol{v}_{ij}^{*(q)}\boldsymbol{v}_{ij}^{(p)}}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|^2}$$
$$= \left(\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}\right)^2 p_{ij(k)}^{*(q)}\widetilde{p}_{ij(k)}^{(p)}\right) \Big/ \left(\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}p_{ij(k)}^{*(q)}\right)^2\right).$$

To reduce the deviation degree between the PLTS $PL_{ij}^{(q)}(p)$ and its normalization form $\widetilde{PL}_{ij}^{*(q)}(p)$ and obtain the optimal normalized PLTS, the values of $\mathrm{Prj}_{\boldsymbol{v}_{ij}^{(p)}}\left(\boldsymbol{v}_{ij}^{*(q)}\right)$ and $\frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\left(\boldsymbol{v}_{ij}^{(p)}\right)$ should be smaller as much as possible. Based on this idea, two projection-based normalization models (NLP1) and (NLP2) are established as follows to normalize the incomplete PLTS via maximizing $\mathrm{Prj}_{\boldsymbol{v}_{ij}^{(p)}}\left(\boldsymbol{v}_{ij}^{*(q)}\right)$ and $\frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\left(\boldsymbol{v}_{ij}^{(p)}\right)$.

$$(\mathrm{NLP1})\ \max \sum_{p=1}^{Q}\mathrm{Prj}_{\boldsymbol{v}_{ij}^{(p)}}\left(\boldsymbol{v}_{ij}^{*(q)}\right)$$
$$= \sum_{p,q=1}^{Q}\left(\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}\right)^2 p_{ij(k)}^{*(q)}\widetilde{p}_{ij(k)}^{(p)}\right)\Big/\sqrt{\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}\widetilde{p}_{ij(k)}^{(p)}\right)^2}$$

$$\text{s.t.}\begin{cases}\sum\limits_{k=1}^{\delta+1} p_{ij(k)}^{*(q)} = 1,\\ \widetilde{p}_{ij(k)}^{(q)} \le p_{ij(k)}^{*(q)} \le 1, k = 1, 2, \ldots, \delta+1.\end{cases}$$

$$(\mathrm{NLP2})\ \max \sum_{p=1}^{Q}\frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\left(\boldsymbol{v}_{ij}^{(p)}\right)$$
$$= \sum_{p,q=1}^{Q}\left(\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}\right)^2 p_{ij(k)}^{*(q)}\widetilde{p}_{ij(k)}^{(p)}\right)\Big/\left(\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}p_{ij(k)}^{*(q)}\right)^2\right)$$

$$\text{s.t.} \begin{cases} \sum_{k=1}^{\delta+1} p_{ij(k)}^{*(q)} = 1, \\ \widetilde{p}_{ij(k)}^{(q)} \le p_{ij(k)}^{*(q)} \le 1, k = 1, 2, \ldots, \delta+1. \end{cases} \tag{1}$$

Then the probability sets $P_{ij}^{1(q)} = \left\{p_{1(1)}^{*(q)}, p_{1(2)}^{*(q)}, \ldots, p_{1(\delta+1)}^{*(q)}\right\}$ and $P_{ij}^{2(q)} = \left\{p_{2(1)}^{*(q)}, p_{2(2)}^{*(q)}, \ldots, p_{2(\delta+1)}^{*(q)}\right\}$ of the normalized PLTSs are obtained via solving the projection-based normalization models (NLP1) and (NLP2), respectively.

However, sometimes $P_{ij}^{1(q)} \neq P_{ij}^{2(q)}$, i.e., we may obtain two unequal normalized PLTSs about one PLTS. To increase the availability and comparability of the normalized PLTSs obtained from the projection-based normalization models (NLP1) and (NLP2) respectively, we build a single objective non-linear optimization model, named projection-based normalization model (NLP3), via merging (NLP1) with (NLP2) to consider the $\mathrm{Prj}_{\boldsymbol{v}_{ij}^{(p)}}\left(\boldsymbol{v}_{ij}^{*(q)}\right)$ and $\frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\left(\boldsymbol{v}_{ij}^{(p)}\right)$ simultaneously.

$$\text{(NLP3)}\ \ \max \sum_{p=1}^{Q}\left(\mathrm{Prj}_{\boldsymbol{v}_{ij}^{(p)}}\left(\boldsymbol{v}_{ij}^{*(q)}\right) + \frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{\boldsymbol{v}_{ij}^{*(q)}}\left(\boldsymbol{v}_{ij}^{(p)}\right)\right)$$

$$\text{s.t.} \begin{cases} \sum_{k=1}^{\delta+1} p_{ij(k)}^{*(q)} = 1, \\ \widetilde{p}_{ij(k)}^{(q)} \le p_{ij(k)}^{*(q)} \le 1, k = 1, 2, \ldots, \delta+1. \end{cases}$$

Solving the normalization model (NLP3), we obtain the probability set of the PLTS $\widetilde{PL}_{ij}^{*(q)}(p)$ is $P_{ij}^{(q)} = \left\{p_{(1)}^{*(q)}, p_{(2)}^{*(q)}, \ldots, p_{(\delta+1)}^{*(q)}\right\}$, where $\widetilde{PL}_{ij}^{*(q)}(p)$ is the normalization of the PLTS $PL_{ij}^{(q)}(p)$. Finally, the normalized $\widetilde{PL}_{ij}^{*(q)}(p)$ can be determined according to the projection-based normalization model (NLP3).

Compared with the normalization method proposed in Pang et al. (2016), the following example is given to indicate that the projection-based normalization model (NLP3) is more rational and the normalized result is more accurate.

*Example 2* Suppose that three DMs are invited to assess the robustness of the LSB technique via using the LTS $\widetilde{L} = \{l_0 = \textit{very bad},\ l_1 = \textit{bad},\ l_2 = \textit{rather poor},\ l_3 = \textit{fairly good},\ l_4 = \textit{good},\ l_5 = \textit{very good},\ l_6 = \textit{excellent}\}$, and their assessment results are $PL^{(1)}(p) = \{l_2(0.5), l_3(0.2)\}$, $PL^{(2)}(p) = \{l_1(0.3), l_2(0.4)\}$, $PL^{(3)}(p) = \{l_2(0.4), l_3(0.1)\}$. In the following, take the solution of the normalized PLTS $PL^{(1)}(p)$ as an example.

(1) According to Definitions 3, the equivalent expression forms of the PLTSs $PL^{(1)}(p)$, $PL^{(2)}(p)$ and $PL^{(3)}(p)$ are

$\widetilde{PL}^{(1)}(p) = \{l_0(0), l_1(0), l_2(0.5), l_3(0.2), l_4(0), l_5(0), l_6(0)\}$, $\widetilde{PL}^{(2)}(p) = \{l_0(0), l_1(0.3), l_2(0.4), l_3(0), l_4(0), l_5(0), l_6(0)\}$,

$\widetilde{PL}^{(3)}(p) = \{l_0(0), l_1(0), l_2(0.4), l_3(0.1), l_4(0), l_5(0), l_6(0)\}$.

(2) Let PLTS $\widetilde{PL}^{*(1)}(p)$ be the normalization of the PLTS $PL^{(1)}(p)$, then via Definitions 5 and 10, the associated vectors of the PLTSs $\widetilde{PL}^{*(1)}(p)$, $\widetilde{PL}^{(2)}(p)$ and $\widetilde{PL}^{(3)}(p)$ are $\boldsymbol{v}^{*(1)} = \left(\frac{1}{7}p^{*(1)}_{(1)}, \frac{2}{7}p^{*(1)}_{(2)}, \frac{3}{7} \times 0.5 \times p^{*(1)}_{(3)}, \frac{4}{7} \times 0.4 \times p^{*(1)}_{(4)}, \frac{5}{7}p^{*(1)}_{(5)}, \frac{6}{7}p^{*(1)}_{(6)}, p^{*(1)}_{(7)}\right)$, $\boldsymbol{v}^{(2)} = \left(0, \frac{2}{7} \times 0.3, \frac{3}{7} \times 0.4, 0, 0, 0, 0\right)$ and $\boldsymbol{v}^{(3)} = \left(0, 0, \frac{3}{7} \times 0.4, \frac{4}{7} \times 0.1, 0, 0, 0\right)$.

(3) By solving the normalization model (NLP3), the probability set of the PLTS $\widetilde{PL}^{*(1)}(p)$ is $P = \{0.1, 0.2, 0.5, 0.2, 0, 0, 0\}$, thus the normalized PLTS $PL^{(1)}(p)$ is $\widetilde{PL}^{*(1)}(p) = \{l_0(0.1), l_1(0.2), l_2(0.5), l_3(0.2)\}$.

Similarly, the normalization of the PLTSs $PL^{(2)}(p)$ and $PL^{(3)}(p)$ are $\widetilde{PL}^{*(2)}(p) = \{l_0(0.2), l_1(0.3), l_2(0.4), l_3(0.1)\}$ and $\widetilde{PL}^{*(3)}(p) = \{l_0(0.3), l_1(0.2), l_2(0.4), l_3(0.1)\}$, respectively.

In addition, according to the normalization method (Pang et al. 2016), the normalization of the PLTSs $PL^{(1)}(p), PL^{(2)}(p)$ and $PL^{(3)}(p)$ are $PL^{*(1)}(p) = \{l_2(0.71), l_3(0.29)\}$, $PL^{*(2)}(p) = \{l_1(0.43), l_2(0.57)\}$, $PL^{*(3)}(p) = \{l_2(0.8), l_3(0.2)\}$. And via Definition 9, we obtain the deviation degree $d\left(PL^{(1)}(p), \widetilde{PL}^{*(1)}(p)\right) = 0.100 < d\left(PL^{(1)}(p), PL^{*(1)}(p)\right) = 0.168, d\left(PL^{(2)}(p), \widetilde{PL}^{*(2)}(p)\right) = 0.087 < d\left(PL^{(2)}(p), PL^{*(2)}(p)\right) = 0.137, d\left(PL^{(3)}(p), \widetilde{PL}^{*(3)}(p)\right) = 0.100 < d\left(PL^{(3)}(p), PL^{*(3)}(p)\right) = 0.296$.

Therefore, the projection-based normalization model (NLP3) is more reasonable and accurate than the normalization method proposed in Pang et al. (2016).

From the above, in the normalization processes, it is obvious that whether to consider the possibilities of the remaining linguistic terms in LTS except those that appear in PLTS will result in different normalized results, which will have a certain impact on the accuracy of the decision-making. As we can see from above, the projection-based normalization model presented in this paper has the following advantages:

(1) The probability of each linguistic term in the normalized PLTS will not reduce so that the original probabilistic linguistic assessment information given by the DMs can be well preserved. In other words, when using the projection-based normalization model for normalization, the DM's willingness is maximally unchanged.

(2) The projection-based normalization model can consider the probabilities of the remaining linguistic terms in LTS except those that appear in PLTS, which can effectively make up for the lack of information. Therefore, all the linguistic terms in the LTS can be treated relatively fairly in the normalization processes.

(3) In the normalization processes, since the projection-based normalization model can simultaneously consider the module sizes and the directions of the associated vectors, the normalized results obtained by this model are more consistent with the real assessments provided by the DMs.

From Example 2, we find that the deviation degree between the PLTSs $PL^{(q)}(p)$ and $\widetilde{PL}^{*(q)}(p)$ is less than the deviation degree between the PLTSs $PL^{(q)}(p)$ and $PL^{*(q)}(p) (q = 1, 2, 3)$. That is to say, the individual similarity degree between the PLTS and its normalization which is obtained by the projection-based normalization model (NLP3) is no less than the individual similarity degree between the PLTS and its normalization which is derived by the normalization method proposed in Pang et al. (2016).

(4) Besides, the normalized results derived by this model can maximize the group similarity degree under the premise that all individual similarity degrees reach the maximum.

*Remark 4* To improve the flexibility of the normalization, corresponding normalization models should be proposed for different types of the uncertain PL-MCDM problems. Based on the LTS $\widetilde{L} = \{l_0, l_1, \ldots, l_\delta\}$ or $\widehat{L} = \{l_t \mid t = -\delta, \ldots, -1, 0, 1, \ldots, \delta\}$, three normalization models are proposed for different types of the uncertain PL-MCDM problems as follows.

(I) The projection-based normalization model of the PLTSs on LTS $\widehat{L}$ in the uncertain multiple DMs PL-MCDM problems

The formulas of the $\mathrm{Prj}_{v_{ij}^{(p)}}(\boldsymbol{v}_{ij}^{*(q)})$ and $\frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{v_{ij}^{*(q)}}(\boldsymbol{v}_{ij}^{(p)})$ are given as:

$$\mathrm{Prj}_{v_{ij}^{(p)}}(\boldsymbol{v}_{ij}^{*(q)}) = \left(\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\right)^2 p_{ij(k)}^{*(q)}\widehat{p}_{ij(k)}^{(p)}\right)\Big/\sqrt{\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\widehat{p}_{ij(k)}^{(p)}\right)^2},$$

$$\frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{v_{ij}^{*(q)}}(\boldsymbol{v}_{ij}^{(p)}) = \left(\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\right)^2 p_{ij(k)}^{*(q)}\widehat{p}_{ij(k)}^{(p)}\right)\Big/ \left(\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}p_{ij(k)}^{*(q)}\right)^2\right).$$

Then the projection-based normalization model (NLP4) is built as follows:

$$(\text{NLP4})\quad \max \sum_{p=1}^{Q}\left(\mathrm{Prj}_{v_{ij}^{(p)}}(\boldsymbol{v}_{ij}^{*(q)}) + \frac{1}{\left|\boldsymbol{v}_{ij}^{*(q)}\right|}\mathrm{Prj}_{v_{ij}^{*(q)}}(\boldsymbol{v}_{ij}^{(p)})\right)$$

$$\text{s.t.}\begin{cases}\sum_{k=1}^{2\delta+1} p_{ij(k)}^{*(q)} = 1,\\ \widehat{p}_{ij(k)}^{(q)} \le p_{ij(k)}^{*(q)} \le 1, k = 1, 2, \ldots, 2\delta+1.\end{cases}$$

(II) The projection-based normalization models of the PLTSs in the uncertain single DM PL-MCDM problems

If the number of the DMs is 1, i.e., $Q = 1$ and $p = q$, then the uncertain multiple DMs PL-MCDM problems reduce to the uncertain single DM PL-MCDM problems. Therefore, the projection-based normalization models of the PLTS on LTSs $\widetilde{L}$ and

$\widehat{L}$ in the uncertain single DM PL-MCDM problems can be derived from the models (NLP3) and (NLP4) respectively as follows:

(i) The projection-based normalization model of the PLTS on LTS $\widetilde{L}$

$$\text{(NLP5)}\ \max \left(\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}\right)^2 p^*_{(k)}\widetilde{p}_{(k)}\right)\left(1\Bigg/\sqrt{\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}\widetilde{p}_{(k)}\right)^2}+1\Bigg/\left(\sum_{k=1}^{\delta+1}\left(\frac{k}{\delta+1}p^*_{(k)}\right)^2\right)\right)$$

$$\text{s.t.}\begin{cases}\sum\limits_{k=1}^{\delta+1} p^*_{(k)}=1,\\ \widetilde{p}_{(k)}\le p^*_{(k)}\le 1, k=1,2,\ldots,\delta+1.\end{cases}$$

(ii) The projection-based normalization model of the PLTS on LTS $\widehat{L}$

$$\text{(NLP6)}\ \max \left(\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\right)^2 p^*_{(k)}\widehat{p}_{(k)}\right)\left(1\Bigg/\sqrt{\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}\widehat{p}_{(k)}\right)^2}+1\Bigg/\left(\sum_{k=1}^{2\delta+1}\left(\frac{2k-2\delta-3}{4\delta+2}p^*_{(k)}\right)^2\right)\right)$$

$$\text{s.t.}\begin{cases}\sum\limits_{k=1}^{2\delta+1} p^*_{(k)}=1,\\ \widehat{p}_{(k)}\le p^*_{(k)}\le 1, k=1,2,\ldots,2\delta+1.\end{cases}$$

### 4.3 The probabilistic linguistic two-step method

For the uncertain multiple DMs PL-MCDM problem with the incomplete PLTS on LTS $\widetilde{L}$, we have presented the projection-based normalization model (NLP3) to overcome the shortages of the existing normalization methods (Pang et al. 2016; Ma et al. 2018). In what follows, we apply the probabilistic linguistic two-step method to solve the uncertain multiple DMs PL-MCDM problems where the criteria weights values are not precisely known but the ranges are available.

By solving the projection-based normalization model (NLP3), the normalized probabilistic linguistic decision matrixes of the matrixes $\boldsymbol{PL}^{(1)}, \boldsymbol{PL}^{(2)}, \ldots, \boldsymbol{PL}^{(Q)}$ can be obtained, denoted as $\boldsymbol{PL}^{*(q)}=\left(PL_{ij}^{*(q)}(p)\right)_{m\times n}$ $(q=1,2,\ldots,Q)$.

Below we convert these normalized probabilistic linguistic decision matrixes into a real value matrix.

For the normalized probabilistic linguistic decision matrix $\boldsymbol{PL^{*(q)}}$ $(q = 1, 2, \ldots, Q)$, the probabilistic linguistic group decision matrix $\boldsymbol{PL^*} = \left(PL^*_{ij}(p)\right)_{m\times n}$ can be obtained via Definition 2. Then the probabilistic linguistic vector matrix $\boldsymbol{V} = (\boldsymbol{v^*_{ij}})_{m\times n}$ which is composed of the associated vector $\boldsymbol{v^*_{ij}}$ of each PLTS $PL^*_{ij}(p)$ in the matrix $\boldsymbol{PL^*}$, and the probabilistic linguistic real value matrix $\boldsymbol{R} = (r_{ij})_{m\times n}$ where $r_{ij} = |\boldsymbol{v^*_{ij}}| (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ can be determined. Therefore, the comprehensive criterion value of the alternative $\psi_j (j = 1, 2, \ldots, n)$ under the probabilistic linguistic circumstance is $z_j = \sum_{i=1}^{m} \omega_i r_{ij}$, where $\sum_{i=1}^{m} \omega_i = 1$. And the larger the comprehensive criterion value is, the better the alternative will be. With the idea of the classical two-step method, the algorithm of the probabilistic linguistic two-step method can be summarized as follows:

**Algorithm 2: (The probabilistic linguistic two-step method)**

**Step 1.** Normalize the probabilistic linguistic decision matrixes $\boldsymbol{PL^{(1)}}, \boldsymbol{PL^{(2)}}, \ldots, \boldsymbol{PL^{(Q)}}$ via solving the projection-based normalization model (NLP3);

**Step 2.** Integrate the probabilistic linguistic decision matrixes provided by each DM to a probabilistic linguistic group decision matrix $\boldsymbol{PL^*}$;

**Step 3.** Determine the probabilistic linguistic real value matrix $\boldsymbol{R} = (r_{ij})_{m\times n}$, where $r_{ij} = |\boldsymbol{v^*_{ij}}|$, and $\boldsymbol{v^*_{ij}}$ is the associated vector of the PLTS $PL^*_{ij}(p)$ in the matrix $\boldsymbol{PL^*}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$;

**Step 4.** Determine the optimal criterion weight vector of each alternative via solving the model (LP1), and then the weight matrix $\boldsymbol{W} = (\boldsymbol{\Omega_1}, \boldsymbol{\Omega_2}, \ldots, \boldsymbol{\Omega_n}) = \left(\omega_i^j\right)_{m\times n}$ can be obtained;

**Step 5.** Calculate the eigenvector $\boldsymbol{\eta}$ corresponding to the maximum eigenvalue of the matrix $(\boldsymbol{W}^\top \boldsymbol{R})(\boldsymbol{W}^\top \boldsymbol{R})^\top$, and then determine the normalized combined weight vector $\boldsymbol{\Omega^*} = \boldsymbol{W}\boldsymbol{\eta} = (\omega_1^*, \omega_2^*, \ldots, \omega_m^*)^\top$ where $\sum_{i=1}^{m} \omega_i^* = 1$.

**Step 6.** Sort the alternatives by the value of $z_j^*$ via formula $z_j^* = \sum_{i=1}^{m} \omega_i^* r_{ij} (j = 1, 2, \ldots, n)$;

**Step 7.** End.

## 5 Illustrative example: the assessment of the data hiding techniques

In this section, we use a case about the performance assessment of the data hiding techniques in a given situation to illustrate the feasibility of the projection-based normalization model and the probabilistic linguistic two-step method.

Suppose there are three DMs $d^{(1)}, d^{(2)}$ and $d^{(3)}$ with the same importance are invited to form a group to provide their assessment information on four data hiding techniques: $\psi_1$ (Least Significant Bits, LSB), $\psi_2$ (Discrete Wavelet Transform, DWT), $\psi_3$ (Data Hiding by Template ranking with symmetrical Central pixels, DHTC), $\psi_4$ (Pair-Wise Logical Computation, PWLC) with respect to five criteria: $\theta_1$ (imperceptibility), $\theta_2$ (undetectability), $\theta_3$ (robustness), $\theta_4$ (security), $\theta_5$ (self-restoring), where the criteria weight vector $\boldsymbol{\Omega} = \left\{(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)^\top \,\middle|\, 0.1 \le \omega_1 \le 0.3,\ 0.1 \le \omega_2 \le 0.2,\ 0.2 \le \omega_3 \le 0.4,\ 0.25 \le \omega_4 \le 0.5,\ 0.01 \le \omega_5 \le 0.2, \sum_{i=1}^{5} \omega_i = 1\right\}$. And the assessment information of the data hiding technique $\psi_j$ with respect to the criterion $\theta_i$ given by DM $d^{(q)}(q = 1, 2, 3)$ is denoted as one PLTS $PL_{ij}^{(q)}(p)(i = 1, 2, 3, 4, 5; j = 1, 2, 3, 4; q = 1, 2, 3)$ via using the LTS $\widetilde{L} = \{l_0 = very\ bad,\ l_1 = bad,\ l_2 = rather\ poor,\ l_3 = fairly\ good,\ l_4 = good,\ l_5 = very\ good,\ l_6 = excellent\}$. Then the probabilistic linguistic decision matrixes $\boldsymbol{PL^{(q)}}$ $(q = 1, 2, 3)$ provided by these three DMs are shown in Tables 1, 2 and Table 3.

**Step 1.** Normalize these three probabilistic linguistic decision matrixes $\boldsymbol{PL^{(1)}}$, $\boldsymbol{PL^{(2)}}$ and $\boldsymbol{PL^{(3)}}$ via solving the projection-based normalization model (NLP3), the normalized results are listed in Tables 4, 5 and 6.

**Step 2.** Integrate all the normalized probabilistic linguistic decision matrixes $\boldsymbol{PL_1^{*(1)}}$, $\boldsymbol{PL_1^{*(2)}}$ and $\boldsymbol{PL_1^{*(3)}}$ to a normalized probabilistic linguistic group decision matrix $\boldsymbol{PL_1}$ via Definition 2, the results of the integration are shown in Table 7.

**Step 3.** Determine the probabilistic linguistic real value matrix $\boldsymbol{R_1}$.

$$\boldsymbol{R_1} = (r_{ij})_{5\times 4} = \begin{pmatrix} 0.211 & 0.484 & 0.227 & 0.391 \\ 0.484 & 0.190 & 0.196 & 0.266 \\ 0.299 & 0.255 & 0.299 & 0.399 \\ 0.348 & 0.372 & 0.193 & 0.193 \\ 0.181 & 0.222 & 0.381 & 0.172 \end{pmatrix},$$

where $r_{ij} = \left|\boldsymbol{v_{ij}^*}\right|$, and $\boldsymbol{v_{ij}^*}$ is the associated vector of the PLTS $PL_{ij}^*(p)$ in the matrix $\boldsymbol{PL_1^*}$ $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3, 4)$.

**Step 4.** By solving the model (LP1), the optimal criterion weight vectors of all alternatives are $\boldsymbol{\Omega}_1 = (0.10, 0.20, 0.20, 0.49, 0.01)^\top$, $\boldsymbol{\Omega}_2 = (0.30, 0.10, 0.20, 0.39, 0.01)^\top$, $\boldsymbol{\Omega}_3 = (0.10, 0.10, 0.35, 0.25, 0.20)^\top$ and $\boldsymbol{\Omega}_4 = (0.24, 0.10, 0.40, 0.25, 0.01)^\top$. Then the weight matrix is

$$\boldsymbol{W_1} = \left(\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3, \boldsymbol{\Omega}_4\right) = \begin{pmatrix} 0.10 & 0.30 & 0.10 & 0.24 \\ 0.20 & 0.10 & 0.10 & 0.10 \\ 0.20 & 0.20 & 0.35 & 0.40 \\ 0.49 & 0.39 & 0.25 & 0.25 \\ 0.01 & 0.01 & 0.20 & 0.01 \end{pmatrix}.$$

**Step 5.** Calculate the eigenvector $\boldsymbol{\eta_1}$ corresponding to the maximum eigenvalue of the matrix $(\boldsymbol{W_1}^\top \boldsymbol{R_1})(\boldsymbol{W_1}^\top \boldsymbol{R_1})^\top$. Using the software MATLAB R2014b, we obtain $\boldsymbol{\eta_1} = (0.493, 0.510, 0.483, 0.514)^\top$ satisfied $\boldsymbol{\eta_1}^\top \boldsymbol{\eta_1} = 1$, and the normalized combined weight vector is $\boldsymbol{\Omega_1^*} = (0.187, 0.125, 0.288, 0.345, 0.056)^\top$.

**Step 6.** Sort the data hiding techniques by the value of $z_j^*(j = 1, 2, 3, 4)$. According to formula $z_j^* = \sum_{i=1}^m \omega_i^* r_{ij}$ where $\omega_i^* \in \boldsymbol{\Omega_1^*}$, we obtain $z_2^* = 0.329 > z_1^* = 0.316 > z_4^* = 0.298 > z_3^* = 0.241$. Thus $\psi_2$ (Discrete Wavelet Transform, DWT) is the best data hiding technique for the given situation.

**Table 1** The probabilistic linguistic decision matrix $\boldsymbol{PL}^{(\mathbf{1})}$ provided by DM $d^{(1)}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_2(0.5), l_3(0.2)\}$ | $\{l_4(0.8), l_6(0.2)\}$ | $\{l_2(0.4), l_3(0.2)\}$ | $\{l_3(0.4), l_5(0.6)\}$ |
| $\theta_2$ | $\{l_3(0.4), l_5(0.6)\}$ | $\{l_0(0.1), l_1(0.5)\}$ | $\{l_0(0.1), l_2(0.2), l_3(0.3)\}$ | $\{l_3(0.6)\}$ |
| $\theta_3$ | $\{l_1(0.2), l_2(0.4), l_3(0.2)\}$ | $\{l_2(0.8), l_3(0.2)\}$ | $\{l_1(0.1), l_3(0.2)\}$ | $\{l_3(0.8), l_5(0.2)\}$ |
| $\theta_4$ | $\{l_3(0.6), l_4(0.4)\}$ | $\{l_1(0.1), l_2(0.4), l_3(0.4)\}$ | $\{l_1(0.5), l_2(0.4), l_3(0.1)\}$ | $\{l_1(0.2), l_2(0.3)\}$ |
| $\theta_5$ | $\{l_1(0.5), l_2(0.1), l_3(0.2)\}$ | $\{l_1(0.1), l_2(0.6)\}$ | $\{l_3(0.4), l_4(0.3)\}$ | $\{l_1(0.4)\}$ |

**Table 2** The probabilistic linguistic decision matrix $\boldsymbol{PL}^{(2)}$ provided by DM $d^{(2)}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_1(0.3), l_2(0.4)\}$ | $\{l_3(0.3), l_4(0.7)\}$ | $\{l_1(0.2), l_2(0.4), l_3(0.4)\}$ | $\{l_3(0.2), l_4(0.4)\}$ |
| $\theta_2$ | $\{l_3(0.3), l_4(0.3)\}$ | $\{l_2(0.3)\}$ | $\{l_2(0.3), l_3(0.2)\}$ | $\{l_0(0.2), l_1(0.2), l_2(0.6)\}$ |
| $\theta_3$ | $\{l_2(0.2), l_3(0.5)\}$ | $\{l_1(0.2), l_2(0.5), l_3(0.3)\}$ | $\{l_2(0.4), l_3(0.6)\}$ | $\{l_3(0.5), l_4(0.3)\}$ |
| $\theta_4$ | $\{l_2(0.5), l_4(0.2)\}$ | $\{l_3(0.6), l_4(0.4)\}$ | $\{l_1(0.3), l_2(0.3)\}$ | $\{l_0(0.3), l_1(0.6), l_2(0.1)\}$ |
| $\theta_5$ | $\{l_0(0.1), l_1(0.6), l_2(0.3)\}$ | $\{l_0(0.4), l_2(0.1)\}$ | $\{l_3(0.7), l_4(0.3)\}$ | $\{l_1(0.2), l_2(0.3)\}$ |

**Table 3** The probabilistic linguistic decision matrix $\boldsymbol{PL}^{(3)}$ provided by DM $d^{(3)}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_2(0.4), l_3(0.1)\}$ | $\{l_3(0.1), l_4(0.4), l_5(0.2)\}$ | $\{l_1(0.3), l_2(0.4)\}$ | $\{l_3(0.3), l_4(0.7)\}$ |
| $\theta_2$ | $\{l_4(0.1), l_5(0.5)\}$ | $\{l_1(0.2), l_2(0.3)\}$ | $\{l_1(0.4), l_2(0.2)\}$ | $\{l_2(0.3), l_3(0.4)\}$ |
| $\theta_3$ | $\{l_1(0.2), l_3(0.6)\}$ | $\{l_2(0.3), l_3(0.1)\}$ | $\{l_0(0.2), l_2(0.3), l_3(0.5)\}$ | $\{l_3(0.6), l_4(0.2), l_5(0.2)\}$ |
| $\theta_4$ | $\{l_2(0.3), l_3(0.3), l_4(0.4)\}$ | $\{l_2(0.1), l_3(0.7), l_4(0.2)\}$ | $\{l_0(0.2), l_2(0.1), l_3(0.2)\}$ | $\{l_1(0.6), l_2(0.4)\}$ |
| $\theta_5$ | $\{l_1(0.3), l_2(0.2)\}$ | $\{l_1(0.4), l_2(0.6)\}$ | $\{l_2(0.2), l_3(0.6), l_4(0.2)\}$ | $\{l_0(0.2), l_1(0.5), l_2(0.2)\}$ |

**Table 4** The normalized probabilistic linguistic decision matrix $\boldsymbol{PL_1^{*(1)}}$ of $\boldsymbol{PL^{(1)}}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_0(0.1), l_1(0.2), l_2(0.5), l_3(0.2)\}$ | $\{l_4(0.8), l_6(0.2)\}$ | $\{l_0(0.2), l_1(0.2), l_2(0.4), l_3(0.2)\}$ | $\{l_3(0.4), l_5(0.6)\}$ |
| $\theta_2$ | $\{l_3(0.4), l_5(0.6)\}$ | $\{l_0(0.1), l_1(0.5), l_2(0.4)\}$ | $\{l_0(0.1), l_1(0.3), l_2(0.3), l_3(0.3)\}$ | $\{l_2(0.4), l_3(0.6)\}$ |
| $\theta_3$ | $\{l_1(0.2), l_2(0.4), l_3(0.3), l_5(0.1)\}$ | $\{l_2(0.8), l_3(0.2)\}$ | $\{l_0(0.4), l_1(0.1), l_2(0.2), l_3(0.3)\}$ | $\{l_3(0.8), l_5(0.2)\}$ |
| $\theta_4$ | $\{l_3(0.6), l_4(0.4)\}$ | $\{l_0(0.1), l_2(0.4), l_3(0.4), l_4(0.1)\}$ | $\{l_1(0.5), l_2(0.4), l_3(0.1)\}$ | $\{l_0(0.3), l_1(0.4), l_2(0.3)\}$ |
| $\theta_5$ | $\{l_1(0.5), l_2(0.2), l_3(0.2), l_4(0.1)\}$ | $\{l_0(0.2), l_1(0.2), l_2(0.6)\}$ | $\{l_0(0.2), l_2(0.1), l_3(0.4), l_4(0.3)\}$ | $\{l_0(0.4), l_1(0.4), l_2(0.2)\}$ |

**Table 5** The normalized probabilistic linguistic decision matrix $\boldsymbol{PL_1^{*(2)}}$ of $\boldsymbol{PL^{(2)}}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_0(0.2), l_1(0.3), l_2(0.4), l_3(0.1)\}$ | $\{l_3(0.3), l_4(0.7)\}$ | $\{l_1(0.2), l_2(0.4), l_3(0.4)\}$ | $\{l_0(0.2), l_3(0.2), l_4(0.4), l_5(0.2)\}$ |
| $\theta_2$ | $\{l_3(0.3), l_4(0.3), l_5(0.4)\}$ | $\{l_0(0.3), l_1(0.4), l_2(0.3)\}$ | $\{l_0(0.3), l_1(0.2), l_2(0.3), l_3(0.2)\}$ | $\{l_0(0.2), l_1(0.2), l_2(0.6)\}$ |
| $\theta_3$ | $\{l_0(0.1), l_1(0.2), l_2(0.2), l_3(0.5)\}$ | $\{l_1(0.2), l_2(0.5), l_3(0.3)\}$ | $\{l_2(0.4), l_3(0.6)\}$ | $\{l_0(0.2), l_3(0.5), l_4(0.2), l_5(0.1)\}$ |
| $\theta_4$ | $\{l_2(0.5), l_3(0.3), l_4(0.2)\}$ | $\{l_3(0.6), l_4(0.4)\}$ | $\{l_0(0.3), l_1(0.3), l_2(0.3), l_3(0.1)\}$ | $\{l_0(0.3), l_1(0.6), l_2(0.1)\}$ |
| $\theta_5$ | $\{l_0(0.1), l_1(0.6), l_2(0.3)\}$ | $\{l_0(0.5), l_1(0.2), l_2(0.3)\}$ | $\{l_3(0.7), l_4(0.3)\}$ | $\{l_0(0.2), l_1(0.5), l_2(0.3)\}$ |

**Table 6** The normalized probabilistic linguistic decision matrix $\boldsymbol{PL_1^{*(3)}}$ of $\boldsymbol{PL^{(3)}}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_0(0.3), l_1(0.2), l_2(0.4), l_3(0.1)\}$ | $\{l_3(0.1), l_4(0.4), l_5(0.5)\}$ | $\{l_0(0.2), l_1(0.3), l_2(0.4), l_3(0.1)\}$ | $\{l_3(0.3), l_4(0.7)\}$ |
| $\theta_2$ | $\{l_3(0.3), l_4(0.2), l_5(0.5)\}$ | $\{l_0(0.3), l_1(0.4), l_2(0.3)\}$ | $\{l_0(0.4), l_1(0.1), l_2(0.2), l_3(0.2)\}$ | $\{l_0(0.2), l_1(0.1), l_2(0.3), l_3(0.4)\}$ |
| $\theta_3$ | $\{l_1(0.2), l_2(0.2), l_3(0.6)\}$ | $\{l_0(0.3), l_1(0.1), l_2(0.4), l_3(0.2)\}$ | $\{l_0(0.2), l_2(0.3), l_3(0.5)\}$ | $\{l_3(0.6), l_4(0.2), l_5(0.2)\}$ |
| $\theta_4$ | $\{l_2(0.3), l_3(0.3), l_4(0.4)\}$ | $\{l_2(0.1), l_3(0.7), l_4(0.2)\}$ | $\{l_0(0.2), l_1(0.3), l_2(0.3), l_3(0.2)\}$ | $\{l_1(0.6), l_2(0.4)\}$ |
| $\theta_5$ | $\{l_0(0.4), l_1(0.4), l_2(0.2)\}$ | $\{l_1(0.4), l_2(0.6)\}$ | $\{l_2(0.2), l_3(0.6), l_4(0.2)\}$ | $\{l_0(0.3), l_1(0.5), l_3(0.2)\}$ |

**Table 7** The normalized probabilistic linguistic group decision matrix $\boldsymbol{PL}_1^*$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_0(0.2), l_1(0.23), l_2(0.44), l_3(0.13)\}$ | $\{l_3(0.13), l_4(0.63), l_5(0.17), l_6(0.07)\}$ | $\{l_0(0.13), l_1(0.23), l_2(0.41), l_3(0.23)\}$ | $\{l_0(0.06), l_3(0.3), l_4(0.37), l_5(0.27)\}$ |
| $\theta_2$ | $\{l_3(0.33), l_4(0.17), l_5(0.5)\}$ | $\{l_0(0.23), l_1(0.44), l_2(0.33)\}$ | $\{l_0(0.27), l_1(0.2), l_2(0.3), l_3(0.23)\}$ | $\{l_0(0.13), l_1(0.1), l_2(0.44), l_3(0.33)\}$ |
| $\theta_3$ | $\{l_0(0.03), l_1(0.2), l_2(0.27), l_3(0.47), l_5(0.03)\}$ | $\{l_0(0.1), l_1(0.1), l_2(0.57), l_3(0.23)\}$ | $\{l_0(0.2), l_1(0.03), l_2(0.3), l_3(0.47)\}$ | $\{l_0(0.07), l_3(0.63), l_4(0.13), l_5(0.17)\}$ |
| $\theta_4$ | $\{l_2(0.27), l_3(0.4), l_4(0.33)\}$ | $\{l_0(0.03), l_2(0.17), l_3(0.57), l_4(0.23)\}$ | $\{l_0(0.17), l_1(0.37), l_2(0.33), l_3(0.13)\}$ | $\{l_0(0.2), l_1(0.53), l_2(0.27)\}$ |
| $\theta_5$ | $\{l_0(0.17), l_1(0.5), l_2(0.23), l_3(0.07), l_4(0.03)\}$ | $\{l_0(0.23), l_1(0.27), l_2(0.5)\}$ | $\{l_0(0.06), l_2(0.1), l_3(0.57), l_4(0.27)\}$ | $\{l_0(0.3), l_1(0.47), l_2(0.23)\}$ |

## 6 Comparative analyses

In this section, we make comparative analyses from two aspects. On the one hand, for the normalization method, to highlight the advantages of the projection-based normalization models proposed in this paper, we adopt the probabilistic linguistic two-stage method based on the normalization method introduced by Pang et al. (2016) to solve the above PL-MCDM problem. On the other hand, to verify the superiority of our proposed the probabilistic linguistic two-step method, the following comparative analyses are among the extended TOPSIS method (Pang et al. 2016), probabilistic linguistic MULTIMOORA method (Wu et al. 2018), and the probabilistic linguistic two-step method proposed by us.

### 6.1 Compared with the existing normalization method

In the following, we employ the probabilistic linguistic two-step method based on the normalization method presented in Pang et al. (2016) to solve the above uncertain multiple DMs PL-MCDM problems for some comparative analyses.

**Step 1.** Normalize the probabilistic linguistic decision matrixes $\boldsymbol{PL}^{(1)}$, $\boldsymbol{PL}^{(2)}$ and $\boldsymbol{PL}^{(3)}$ via using the normalization method (Pang et al. 2016), and the results are listed in Tables 8, 9 and 10.

**Step 2.** According to Definition 2, the normalized probabilistic linguistic group decision matrix $\boldsymbol{PL}_2^*$ is obtained, and the results of the integration are shown in Table 11.

**Step 3.** Determine the probabilistic linguistic real value matrix $\boldsymbol{R}_2$.

$$\boldsymbol{R}_2 = (r_{ij})_{5\times 4} = \begin{pmatrix} 0.312 & 0.512 & 0.279 & 0.418 \\ 0.473 & 0.254 & 0.256 & 0.332 \\ 0.348 & 0.325 & 0.353 & 0.426 \\ 0.305 & 0.372 & 0.209 & 0.222 \\ 0.217 & 0.244 & 0.419 & 0.219 \end{pmatrix},$$

where $r_{ij} = \left|\boldsymbol{v}_{ij}^*\right|$, and $\boldsymbol{v}_{ij}^*$ is the associated vector of the PLTS $PL_{ij}^*(p)$ in the matrix $\boldsymbol{PL}_2^*$ $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3, 4)$.

**Step 4.** Solve the model (LP1) to determine the weight matrix $\boldsymbol{W}_2$, then the result is

$$\boldsymbol{W}_2 = (\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_3, \boldsymbol{\Omega}_4) = \begin{pmatrix} 0.14 & 0.30 & 0.10 & 0.24 \\ 0.20 & 0.10 & 0.10 & 0.10 \\ 0.40 & 0.20 & 0.35 & 0.40 \\ 0.25 & 0.39 & 0.25 & 0.25 \\ 0.01 & 0.01 & 0.20 & 0.01 \end{pmatrix}.$$

**Step 5.** Calculate the eigenvector $\boldsymbol{\eta}_2$ corresponding to the maximum eigenvalue of the matrix $(\boldsymbol{W}_2^\top \boldsymbol{R}_2)(\boldsymbol{W}_2^\top \boldsymbol{R}_2)^\top$. By using the software MATLAB R2014b, we obtain

**Table 8** The normalized probabilistic linguistic decision matrix $\boldsymbol{PL_2^{*(1)}}$ of $\boldsymbol{PL^{(1)}}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_2(0.71), l_3(0.29)\}$ | $\{l_4(0.8), l_6(0.2)\}$ | $\{l_2(0.67), l_3(0.33)\}$ | $\{l_3(0.4), l_5(0.6)\}$ |
| $\theta_2$ | $\{l_3(0.4), l_5(0.6)\}$ | $\{l_0(0.17), l_1(0.83)\}$ | $\{l_0(0.17), l_2(0.33), l_3(0.5)\}$ | $\{l_3(1)\}$ |
| $\theta_3$ | $\{l_1(0.25), l_2(0.5), l_3(0.25)\}$ | $\{l_2(0.8), l_3(0.2)\}$ | $\{l_1(0.33), l_3(0.67)\}$ | $\{l_3(0.8), l_5(0.2)\}$ |
| $\theta_4$ | $\{l_3(0.6), l_4(0.4)\}$ | $\{l_1(0.11), l_2(0.44), l_3(0.44)\}$ | $\{l_1(0.5), l_2(0.4), l_3(0.1)\}$ | $\{l_1(0.33), l_2(0.67)\}$ |
| $\theta_5$ | $\{l_1(0.63), l_2(0.13), l_3(0.25)\}$ | $\{l_1(0.14), l_2(0.86)\}$ | $\{l_3(0.57), l_4(0.43)\}$ | $\{l_1(1)\}$ |

**Table 9** The normalized probabilistic linguistic decision matrix $\boldsymbol{PL_2^{*(2)}}$ of $\boldsymbol{PL^{(2)}}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_1(0.43), l_2(0.57)\}$ | $\{l_3(0.3), l_4(0.7)\}$ | $\{l_1(0.2), l_2(0.4), l_3(0.4)\}$ | $\{l_3(0.33), l_4(0.67)\}$ |
| $\theta_2$ | $\{l_3(0.5), l_4(0.5)\}$ | $\{l_2(1)\}$ | $\{l_2(0.6), l_3(0.4)\}$ | $\{l_0(0.2), l_1(0.2), l_2(0.6)\}$ |
| $\theta_3$ | $\{l_2(0.29), l_3(0.71)\}$ | $\{l_1(0.2), l_2(0.5), l_3(0.3)\}$ | $\{l_2(0.4), l_3(0.6)\}$ | $\{l_3(0.63), l_4(0.38)\}$ |
| $\theta_4$ | $\{l_2(0.71), l_4(0.29)\}$ | $\{l_3(0.6), l_4(0.4)\}$ | $\{l_1(0.5), l_2(0.5)\}$ | $\{l_0(0.3), l_1(0.6), l_2(0.1)\}$ |
| $\theta_5$ | $\{l_0(0.1), l_1(0.6), l_2(0.3)\}$ | $\{l_0(0.8), l_2(0.2)\}$ | $\{l_3(0.7), l_4(0.3)\}$ | $\{l_1(0.4), l_2(0.6)\}$ |

**Table 10** The normalized probabilistic linguistic decision matrix $\boldsymbol{PL_2^{*(3)}}$ of $\boldsymbol{PL^{(3)}}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_2(0.8), l_3(0.2)\}$ | $\{l_3(0.14), l_4(0.57), l_5(0.29)\}$ | $\{l_1(0.43), l_2(0.57)\}$ | $\{l_3(0.3), l_4(0.7)\}$ |
| $\theta_2$ | $\{l_4(0.17), l_5(0.83)\}$ | $\{l_1(0.4), l_2(0.6)\}$ | $\{l_1(0.67), l_2(0.33)\}$ | $\{l_2(0.43), l_3(0.57)\}$ |
| $\theta_3$ | $\{l_1(0.25), l_3(0.75)\}$ | $\{l_2(0.75), l_3(0.25)\}$ | $\{l_0(0.2), l_2(0.3), l_3(0.5)\}$ | $\{l_3(0.6), l_4(0.2), l_5(0.2)\}$ |
| $\theta_4$ | $\{l_2(0.3), l_3(0.3), l_4(0.4)\}$ | $\{l_2(0.1), l_3(0.7), l_4(0.2)\}$ | $\{l_0(0.4), l_2(0.2), l_3(0.4)\}$ | $\{l_1(0.6), l_2(0.4)\}$ |
| $\theta_5$ | $\{l_1(0.6), l_2(0.4)\}$ | $\{l_1(0.4), l_2(0.6)\}$ | $\{l_2(0.2), l_3(0.6), l_4(0.2)\}$ | $\{l_0(0.22), l_1(0.56), l_2(0.22)\}$ |

**Table 11** The normalized probabilistic linguistic group decision matrix $\boldsymbol{PL_2^*}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_1(0.14), l_2(0.69), l_3(0.16)\}$ | $\{l_3(0.15), l_4(0.69), l_5(0.1), l_6(0.07)\}$ | $\{l_1(0.21), l_2(0.55), l_3(0.24)\}$ | $\{l_3(0.34), l_4(0.46), l_5(0.2)\}$ |
| $\theta_2$ | $\{l_3(0.3), l_4(0.22), l_5(0.48)\}$ | $\{l_0(0.06), l_1(0.4), l_2(0.53)\}$ | $\{l_0(0.06), l_1(0.22), l_2(0.42), l_3(0.3)\}$ | $\{l_0(0.07), l_1(0.07), l_2(0.34), l_3(0.52)\}$ |
| $\theta_3$ | $\{l_1(0.17), l_2(0.26), l_3(0.57)\}$ | $\{l_1(0.07), l_2(0.68), l_3(0.25)\}$ | $\{l_0(0.07), l_1(0.11), l_2(0.23), l_3(0.59)\}$ | $\{l_3(0.68), l_4(0.19), l_5(0.13)\}$ |
| $\theta_4$ | $\{l_2(0.34), l_3(0.3), l_4(0.36)\}$ | $\{l_0(0.04), l_2(0.18), l_3(0.58), l_4(0.21)\}$ | $\{l_0(0.13), l_1(0.33), l_2(0.37), l_3(0.17)\}$ | $\{l_0(0.1), l_1(0.51), l_2(0.39)\}$ |
| $\theta_5$ | $\{l_0(0.03), l_1(0.61), l_2(0.28), l_3(0.08)\}$ | $\{l_0(0.27), l_1(0.18), l_2(0.55)\}$ | $\{l_2(0.07), l_3(0.62), l_4(0.31)\}$ | $\{l_0(0.07), l_1(0.65), l_2(0.27)\}$ |

$\boldsymbol{\eta}_2 = (0.505, 0.499, 0.482, 0.514)^\top$ satisfied $\boldsymbol{\eta}_2^\top \boldsymbol{\eta}_2 = 1$, and the normalized combined weight vector is $\boldsymbol{\Omega}_2^* = (0.196, 0.125, 0.338, 0.285, 0.056)^\top$.

**Step 6.** Use the comprehensive criterion value $z_j^*(j = 1, 2, 3, 4)$ to rank the alternatives. After calculation, we obtain $z_2^* = 0.362 > z_4^* = 0.343 > z_1^* = 0.337 > z_3^* = 0.289$. Thus, $\psi_2$ (Discrete Wavelet Transform, DWT) is one of the best.

From above two sorting results derived by the probabilistic linguistic two-step method, we find that the optimal data hiding technique obtained by solving the projection-based normalization model (NLP3) is consistent with the best data hiding technique obtained by using the normalization method introduced in Pang et al. (2016), but the two ranking lists of the specific alternatives are slightly different. The main reason for this situation is that the projection-based normalization model proposed in this paper could consider the probabilities of the remaining linguistic terms in LTS except those that appear in PLTS, which can effectively make up for the lack of information. Therefore, all the linguistic terms in the LTS can be treated relatively fairly in the normalization processes. However, for the PLTS which the sum of the probabilities of all possible linguistic terms is less than 1, the normalization method given by Pang et al. (Pang et al. 2016) is limited to consider the probabilities of the existing linguistic terms in PLTSs, and the probabilities of occurrence of the remaining linguistic terms in the LTS are not taken into account.

Moreover, the normalization model proposed by Ma et al. (2018) has some limitations, for example, its only suit for solving the uncertain multiple DMs PL-MCDM problems and its operations are more complex than any one of the projection-based normalization models (NLP3) and (NLP4). Therefore, the projection-based normalization models proposed in this paper are more reasonable and convincing than the existing normalization approaches (Pang et al. 2016; Ma et al. 2018).

### 6.2 Compared with the extended TOPSIS method

In this section, our proposed the probabilistic linguistic two-step method is compared with the extended TOPSIS method (Pang et al. 2016). When using the extended TOPSIS method to solve the uncertain multiple DMs PL-MCDM problems, the normalized probabilistic linguistic group decision matrix should be determined first, and then determine the criterion weight vector, the PL-PIS, and the PL-NIS. Next, the deviation degrees between each alternative and the PL-PIS/PL-NIS should be calculated. Finally, rank the alternatives through the closeness coefficient $CI$. Now we employ the extended TOPSIS method to handle the above case about the performance assessment of the data hiding techniques.

**Step 1.** According to the algorithm of the extended TOPSIS method introduced in Pang et al. (2016) and Definition 2, the normalized probabilistic linguistic group decision matrix is shown in Table 12.

1

**Step 2.** Calculate the criterion weight vector, and the result is $\boldsymbol{\Omega} = (0.30, 0.20, 0.20, 0.29, 0.01)^\top$.

**Step 3.** Determine the PL-PIS $PL^+(p)$ and the PL-NIS $PL^-(p)$ respectively:

**Table 12** The normalized probabilistic linguistic group decision matrix $\boldsymbol{PL_1^*}$

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | $\{l_0(0), l_0(0.2), l_1(0.23), l_3(0.13), l_2(0.44)\}$ | $\{l_3(0), l_3(0.13), l_6(0.07), l_5(0.17), l_4(0.63)\}$ | $\{l_0(0), l_0(0.13), l_1(0.23), l_3(0.23), l_2(0.41)\}$ | $\{l_0(0), l_0(0.06), l_3(0.3), l_5(0.27), l_4(0.37)\}$ |
| $\theta_2$ | $\{l_0(0), l_0(0), l_4(0.17), l_3(0.33), l_5(0.5)\}$ | $\{l_0(0), l_0(0), l_0(0.23), l_1(0.44), l_2(0.33)\}$ | $\{l_0(0), l_0(0.27), l_1(0.2), l_2(0.3), l_3(0.23)\}$ | $\{l_0(0), l_0(0.13), l_1(0.1), l_2(0.44), l_3(0.33)\}$ |
| $\theta_3$ | $\{l_0(0.03), l_1(0.2), l_2(0.27), l_3(0.47), l_5(0.03)\}$ | $\{l_0(0), l_0(0.1), l_1(0.1), l_3(0.23), l_2(0.57)\}$ | $\{l_0(0), l_0(0.2), l_1(0.03), l_2(0.3), l_3(0.47)\}$ | $\{l_0(0), l_0(0.07), l_4(0.13), l_5(0.17), l_3(0.63)\}$ |
| $\theta_4$ | $\{l_0(0), l_0(0), l_2(0.27), l_3(0.4), l_4(0.33)\}$ | $\{l_0(0), l_0(0.03), l_2(0.17), l_4(0.23), l_3(0.57)\}$ | $\{l_0(0), l_0(0.17), l_1(0.37), l_3(0.13), l_2(0.33)\}$ | $\{l_0(0), l_0(0), l_0(0.2), l_1(0.53), l_2(0.27)\}$ |
| $\theta_5$ | $\{l_0(0.17), l_4(0.03), l_3(0.07), l_1(0.5), l_2(0.23)\}$ | $\{l_0(0), l_0(0), l_0(0.23), l_1(0.27), l_2(0.5)\}$ | $\{l_0(0), l_0(0.06), l_2(0.1), l_4(0.27), l_3(0.57)\}$ | $\{l_0(0), l_0(0), l_0(0.3), l_2(0.23), l_1(0.47)\}$ |

$PL^{+}(p) = \big(\{l_0, l_{0.39}, l_{0.9}, l_{1.35}, l_{2.52}\}, \{l_0, l_0, l_{0.68}, l_{0.99}, l_{2.5}\}, \{l_0, l_{0.15}, l_{0.52}, l_{0.85}, l_{1.89}\}, \{l_0, l_{0.12}, l_{0.21}, l_{1.08}, l_{1.71}\}\big)$,

$PL^{-}(p) = \big(\{l_0, l_0, l_{0.23}, l_{0.39}, l_{0.82}\}, \{l_0, l_0, l_0, l_{0.44}, l_{0.66}\}, \{l_0, l_0, l_{0.1}, l_{0.54}, l_{1.14}\}, \{l_0, l_0, l_0, l_{0.39}, l_{0.54}\}, \{l_0, l_0, l_0, l_{0.27}, l_{0.46}\}\big)$.

**Step 4.** Calculate the deviation degrees between each alternative and the PL-PIS/PL-NIS respectively, then derive the closeness coefficient $CI$ of each alternative. And the results are shown in Table 13.

**Step 5.** Rank the alternatives according to $CI$. According to Table 13, the final rank of the alternatives is $\psi_2 > \psi_1 > \psi_4 > \psi_3$. Thus, the best alternative is $\psi_2$ (Discrete Wavelet Transform, DWT).

As can be seen from the above, the ranking list of the specific alternatives derived by the extended TOPSIS method (Pang et al. 2016) is the same as that obtained by the probabilistic linguistic two-step method presented in this paper. By comparing these two methods, the validity of the proposed probabilistic linguistic two-step method is proved, and the probabilistic linguistic two-step method can be used as a new decision-method method for solving the uncertain PL-MCDM problems, thus enriching the methods to solve the uncertain PL-MCDM problems.

### 6.3 Compared with the probabilistic linguistic MULTIMOORA method

In the following, comparisons between our proposed the probabilistic linguistic two-step method and the probabilistic linguistic MULTIMOORA method (Wu et al. 2018) are conducted. The probabilistic linguistic MULTIMOORA method can solve the uncertain multiple DMs PL-MCDM problems in which the information about the criteria weights is unknown completely. With this method, first of all, we determine the criteria weights and normalize the decision matrix via the expected value function based on the normalized probabilistic linguistic group decision matrix. Then, for three aggregation models, calculate the utility values of each alternative respectively, and then obtain the subordinate rank of each model. Finally, build the normalized utility value matrix, and the final ranks can be obtained by the Borda scores. Now we employ the probabilistic linguistic MULTIMOORA method to handle the above problem.

**Step 1.** Determine the criterion weight vector. Let $PL_j^{+}(p)$ and $PL_j^{-}(p)$ be the best and worst values of criterion $\theta_j$, respectively. Then according to the Hamming distance formula, the values of $d(PL_{ij}(p), PL_j^{+}(p))$ are given as Table 14.

And then the criterion weight vector is $\mathbf{\Omega} = (0.198, 0.171, 0.192, 0.191, 0.248)^{\top}$.

**Step 2.** For the normalized integration results given in Table 7, the vector-normalized values of all alternatives with respect to each criterion can be calculated via the expected value function, and the results are shown in Table 15.

**Step 3.** Based on Tables 14 and 15 and the weights of criteria, the utility values of alternatives and the subordinate ranks derived from PLRS, PLRP, and PLFMF are

**Table 13** The values of the deviation degrees

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ | min or max |
|---|---|---|---|---|---|
| $d(\psi_j, PL^+(p))$ | 0.391 | 0.399 | 0.678 | 0.502 | $d_{\min}(\psi_j, PL^+(p)) = 0.391$ |
| $d(\psi_j, PL^-(p))$ | 0.383 | 0.432 | 0.146 | 0.331 | $d_{\max}(\psi_j, PL^-(p)) = 0.432$ |
| $CI(\psi_j)$ | −0.113 | −0.021 | −1.396 | −0.518 | – |

**Table 14** The vector-normalized values

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | 0.382 | 0 | 0.407 | 0.085 |
| $\theta_2$ | 0 | 0.512 | 0.447 | 0.367 |
| $\theta_3$ | 0.173 | 0.222 | 0.260 | 0 |
| $\theta_4$ | 0 | 0.038 | 0.273 | 0.332 |
| $\theta_5$ | 0.283 | 0.287 | 0 | 0.343 |

**Table 15** The vector-normalized values

| | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_4$ |
|---|---|---|---|---|
| $\theta_1$ | 0.250 | 0.724 | 0.386 | 0.677 |
| $\theta_2$ | 0.696 | 0.193 | 0.330 | 0.357 |
| $\theta_3$ | 0.383 | 0.340 | 0.452 | 0.591 |
| $\theta_4$ | 0.510 | 0.522 | 0.315 | 0.194 |
| $\theta_5$ | 0.215 | 0.224 | 0.662 | 0.169 |

**Table 16** The results derived by the PL-MULTIMOORA method

| | $PLRS$ $u_1(\psi_j)$ | $\text{Rank}_1$ | $PLRP$ $u_2(\psi_j)$ | $\text{Rank}_2$ | $PLFMF$ $u_3(\psi_j)$ | $\text{Rank}_3$ | Borda score | Final rank |
|---|---|---|---|---|---|---|---|---|
| $\psi_1$ | 0.393 | 3 | 0.205 | 1 | 0.092 | 2 | 0.086 | 3 |
| $\psi_2$ | 0.397 | 2 | 0.208 | 2 | 0.087 | 3 | 0.095 | 2 |
| $\psi_3$ | 0.444 | 1 | 0.225 | 3 | 0.102 | 1 | 0.151 | 1 |
| $\psi_4$ | 0.388 | 4 | 0.248 | 4 | 0.084 | 4 | −0.052 | 4 |

obtained. Then the utility value matrix and the rank matrix are established as follows:

$$\boldsymbol{D}(\boldsymbol{u}) = \begin{pmatrix} 0.393 & 0.205 & 0.092 \\ 0.397 & 0.208 & 0.087 \\ 0.444 & 0.225 & 0.102 \\ 0.388 & 0.248 & 0.084 \end{pmatrix}, \quad \boldsymbol{D}(\boldsymbol{r}) = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 3 & 1 \\ 4 & 4 & 4 \end{pmatrix}.$$

And the vector-normalized utility value matrix is:

$$\boldsymbol{D}^{*}(\boldsymbol{u}) = \begin{pmatrix} 0.484 & 0.461 & 0.503 \\ 0.489 & 0.468 & 0.475 \\ 0.547 & 0.506 & 0.557 \\ 0.478 & 0.558 & 0.459 \end{pmatrix}.$$

Based on $\boldsymbol{D}^{*}(\boldsymbol{u})$, the Borda scores of alternatives are $IBS_1 = 0.202$, $IBS_2 = 0.148$, $IBS_3 = 0.290$, $IBS_4 = -0.130$. To simplify representation, the above results are listed in Table 16.

From Table 16, we can see that the data hiding techniques $\psi_3$ (Data Hiding by Template ranking with symmetrical Central pixels, DHTC) is the best one for this problem.

By comparing these two methods, we find that the ordering of the alternatives obtained by the probabilistic linguistic MULTIMOORA method is very different from the ordering obtained by the probabilistic linguistic two-step method. Recalling the above calculation process, we could see that the known partial criteria weight information is not used when determining the criteria weights values, so that the criteria weights values obtained by the probabilistic linguistic MULTIMOORA method do not meet the basic requirements of the known partial criterion weight information, and the final rank of the alternatives will greatly deviate from the real situation. To facilitate comparison, the ranking lists and best solution derived by the above four methods are listed in Table 17.

In summary, the projection-based normalization models proposed based on considering the remaining linguistic terms in LTS except those that appear in PLTS, which can effectively make up for the lack of information, and thus can improve the accuracy of the normalized PLTS. Furthermore, the main idea of the probabilistic linguistic two-step method is to local optimization firstly and then recombination weighting and sort the alternatives. This method not only avoids the difficulty of obtaining preference information, but also utilizes the prior information of the normalized evaluation, which makes the criterion weight information more accurate, and the final evaluation result is more objective and comprehensive. Therefore, the probabilistic linguistic two-step method as a novel decision-making method can enrich the weight-determining method for the uncertain PL-MCDM problems, in which the criteria weights are given in the form of intervals.

## 7 Conclusions

It is reasonable to consider the probabilities of the remaining linguistic terms in LTS except those that appear in PLTS during the normalization process, and the accuracy of the normalization method affects the accuracy of the decision-making result. To improve the accuracy of the normalization method, firstly, we have introduced some novel concepts, such as the equivalent expression forms of the PLTSs, the equivalent transformation functions between the PLTS and its associated vector, the projection formulas of the PLTSs, and a new deviation degree formula. Then, four projection-based normalization models are proposed for different types of uncertain PL-MCDM problems. The projection-based normalization models proposed for the uncertain single DM PL-MCDM problems can not only consider the probabilities of the remaining linguistic terms in LTS except those that appear in PLTS but also have a great individual similarity degree between the PLTS and its normalization. Based on considering the probabilities of the others linguistic terms in LTS except those that appear in PLTS, projection-based normalization models are presented for the uncertain multiples DMs PL-MCDM problems will have a great group similarity degree. Besides, for the uncertain PL-MCDM problems with the criteria weight values not precisely known but the ranges are available, we have introduced the probabilistic linguistic two-step method

**Table 17** The ranking lists and best solutions derived by the above four methods

| Methods | Ranks | Best solutions |
|---|---|---|
| The probabilistic linguistic two-step method based on the normalization method proposed in this paper | $\psi_2 > \psi_1 > \psi_4 > \psi_3$ | $\psi_2$ |
| The probabilistic linguistic two-step method based on the existing normalization method (Pang et al. 2016) | $\psi_2 > \psi_4 > \psi_1 > \psi_3$ | $\psi_2$ |
| The extended TOPSIS method (Pang et al. 2016) | $\psi_2 > \psi_1 > \psi_4 > \psi_3$ | $\psi_2$ |
| The probabilistic linguistic MULTIMOORA method (Wu et al. 2018) | $\psi_3 > \psi_1 > \psi_2 > \psi_4$ | $\psi_3$ |

to determine the criteria weights values for the first time. Finally, we have given a case about the performance assessment of the data hiding techniques available to illustrate the rationality and validity of the projection-based normalization models and the probabilistic linguistic two-step method.

In the future, we will further study the normalization model for the uncertain multiple DMs PL-MCDM problems, in which the assessment information is completely unknown. In addition, based on the projection-based normalization models proposed in this paper, the interaction method that can make full use of the known objective information and maximize the subjective initiative of the DMs will be used to solve PL-MCDM problems.

**Acknowledgements** This work was supported in part by the National Natural Science Foundation of China (No. 11371130), Soft science research program of Fujian Province (No. B19085), the projects of Education Department of Fujian Province (No. JT180263), the open fund of Key Laboratory of Applied Mathematics of Fujian Province University (Putian University) (No. SX201906) and Digital Fujian big data modeling and intelligent computing institute, Pre-Research Fund of Jimei University.

## References

Bai, C. Z., Zhang, R., Shen, S., Huang, C. F., & Fan, X. (2018). Interval-valued probabilistic linguistic term sets in multi-criteria group decision making. *International Journal of Intelligent Systems*, *33*(6), 1301–1321.

Dai, Y. Q., Xu, Z. S., Li, Y., & Da, Q. L. (2008). New evaluation scale of linguistic information and its application. *Chinese Journal of Management Science*, *16*(2), 145–149.

Feng, X. Q., Liu, Q., & Wei, C. P. (2019). Probabilistic linguistic QUALIFLEX approach with possibility degree comparison. *Journal of Intelligent and Fuzzy Systems*, *36*, 719–730.

Gao, J., Xu, Z. S., Ren, P. J., & Liao, H. C. (2019). An emergency decision making method based on the multiplicative consistency of probabilistic linguistic preference relations. *International Journal of Machine Learning and Cybernetics*, *10*(7), 1613–1629.

Gou, X. J., & Xu, Z. S. (2016). Novel basic operational laws for linguistic terms, hesitant fuzzy linguistic term sets and probabilistic linguistic term sets. *Information Sciences*, *372*, 407–427.

Liao, H. C., Jiang, L. S., Xu, Z. S., Xu, J. P., & Herrera, F. (2017). A linear programming method for multiple criteria decision making with probabilistic linguistic information. *Information Sciences*, *415*(416), 341–355.

Liao, H. C., Mi, X. M., & Xu, Z. S. (2019). A survey of decision-making methods with probabilistic linguistic information: bibliometrics, preliminaries, methodologies, applications and future directions. *Fuzzy Optimization and Decision Making*,. https://doi.org/10.1007/s10700-019-09309-5.

Lin, M. W., Chen, Z. Y., Liao, H. C., & Xu, Z. S. (2019). ELECTRE II method to deal with probabilistic linguistic term sets and its application to edge computing. *Nonlinear Dynamics*, *96*(3), 2125–2143.

Lin, M. W., & Xu, Z. S. (2018). Probabilistic linguistic distance measures and their applications in multi-criteria group decision making. *Studies in Fuzziness and Soft Computing*, *357*, 411–440.

Lin, M. W., Xu, Z. S., Zhai, Y. L., & Yao, Z. Q. (2018). Multi-attribute group decision-making under probabilistic uncertain linguistic environment. *Journal of the Operational Research Society*, *69*(2), 157–170.

Liu, P. D., & Teng, F. (2019). Probabilistic linguistic TODIM method for selecting products through online product reviews. *Information Sciences*, *485*, 441–455.

Ma, Z. Z., Zhu, J. J., & Chen, Y. (2018). A probabilistic linguistic group decision-making method from a reliability perspective based on evidential reasoning. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*,. https://doi.org/10.1109/TSMC.2018.2815716.

Pang, Q., Wang, H., & Xu, Z. S. (2016). Probabilistic linguistic term sets in multi-attribute group decision making. *Information Sciences*, *369*, 128–143.

Pan, L., Ren, P. J., & Xu, Z. S. (2018). Therapeutic schedule evaluation for brain-metastasized non-small cell lung cancer with a probabilistic linguistic ELECTRE II method. *International Journal of Environmental Research and Public Health*, *15*(9), 1–23.

Rodríguez, R. M., Martínez, L., & Herrera, F. (2012). Hesitant fuzzy linguistic terms sets for decision making. *IEEE Transactions on Fuzzy Systems*, *20*(1), 109–119.

Song, Y. M., & Hu, J. (2019). Large-scale group decision making with multiple stakeholders based on probabilistic linguistic preference relation. *Applied Soft Computing Journal*, *80*, 721–722.

Wu, X. L., & Liao, H. C. (2018). An approach to quality function deployment based on probabilistic linguistic term sets and ORESTE method for multi-expert multi-criteria decision making. *Information Fusion*, *43*, 13–26.

Wu, X. L., Liao, H. C., Xu, Z. S., Hafezalkotob, A., & Herrera, F. (2018). Probabilistic linguistic MULTIMOORA: a multi-criteria decision making method based on the probabilistic linguistic expectation function and the improved borda rule. *IEEE Transactions on Fuzzy Systems*, *26*, 3688–3702.

Xu, Z. S. (2001). Maximum deviation method based on deviation degree and possibility degree for uncertain multi-attribute decision making. *Control and Decision*, *16*, 818–821.

Xu, Z. S. (2002). On method for multi-objective decision-making with partial weight information. *Systems Engineering-Theory & Practice*, *22*(1), 43–47.

Xu, Z. S. (2004). *Uncertain multiple attribute decision making: Methods and applications*. Berlin: Springer.

Xu, Z. S. (2010). Interactive group decision making procedure based on uncertain multiplicative linguistic preference relations. *Journal of Systems Engineering and Electronics*, *21*(3), 408–415.

Yue, N., Xie, J. L., & Chen, S. L. (2020). Some new basic operations of probabilistic linguistic term sets and their application in multi-criteria decision making. *Soft Computing*,. https://doi.org/10.1007/s00500-019-04651-7.

Zhang, X. F., Gou, X. J., Xu, Z. S., & Liao, H. C. (2018a). A projection method for multiple attribute group decision making with probabilistic linguistic term sets. *International Journal of Machine Learning and Cybernetics*, *10*(9), 2515–2528.

Zhang, Y. X., Xu, Z. S., Wang, H., & Liao, H. C. (2018b). Water security evaluation based on the TODIM method with probabilistic linguistic term sets. *Soft Computing*, *23*(15), 6215–6230.

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.