Perceptual Reasoning Using Interval Type-2 Fuzzy Sets: Properties

Dongrui Wu, *Student Member, IEEE*, and Jerry M. Mendel, *Life Fellow, IEEE*

***Abstract*—Perceptual Reasoning (PR) is an Approximate Reasoning mechanism that can be used as a Computing with Words (CWW) Engine, i.e., given input words, PR can infer the output from a rulebase. When the input words and the words in the rulebase are modeled by interval type-2 fuzzy sets (IT2 FSs), the output of PR, $\tilde{Y}_{PR}$, is also an IT2 FS, and it will be mapped to a word in a codebook. For accurate mapping, we need to ensure that $\tilde{Y}_{PR}$ resembles the IT2 FSs in the codebook. The concept of PR using IT2 FSs was originally proposed in [10]. In this paper, the procedures to compute PR are introduced, and the properties of PR are studied in more detail. More specifically, we show under what conditions $\tilde{Y}_{PR}$ can be a shoulder or interior footprint of uncertainty.**

***Index Terms*—Computing with words, perceptual reasoning, interval type-2 fuzzy sets, linguistic weighted average**

I. Introduction

There are many models for the fuzzy implication, under the rubric of Approximate Reasoning, e.g. Table 11.1 in [3] lists 14. Each of these models has the property that it reduces to the truth table of material implication when fuzziness disappears, and to-date none of these models has been examined using interval type-2 fuzzy sets (IT2 FSs). When Approximate Reasoning is used in Computing with Words (CWW), we do not implement logical reasoning as prescribed by the truth table of material implication; instead we subscribe to rational description [1], which *"is the view that behavior can be approximately described as conforming with the results that would be obtained by some rational calculation."* More specifically, we consider the following problem:

Given a rulebase with N rules, each of the form:

$$R^i: \text{ If } x_1 \text{ is } \tilde{F}_1^i \text{ and } \ldots \text{ and } x_p \text{ is } \tilde{F}_p^i, \text{ Then } y \text{ is } \tilde{G}^i \quad (1)$$

where $\tilde{F}_j^i$ and $\tilde{G}^i$ are words modeled by IT2 FSs (the words and their FOUs constitute a codebook*), and a new input*

$$\tilde{\mathbf{X}}' = (\tilde{X}_1, \ldots, \tilde{X}_p), \quad (2)$$

where $\tilde{X}_i$ are also words (from the codebook) modeled by IT2 FSs, then what is the output IT2 FS $\tilde{Y}$ and its associated word in the codebook?

According to Liu and Mendel's Interval Approach [6], [7] for word modeling [in which interval end-point data are collected from a group of subjects about a word (on a scale of 0-10), and are then mapped into a footprint of uncertainty (FOU)], only three kinds of FOUs emerge, namely left-shoulder, right-shoulder and interior, as shown in Fig. 1. The 32-words vocabulary obtained in [6] are shown in Fig. 2. So, for CWW, when FOUs are synthesized from data, only a very limited number of IT2 FSs can be used to model a rule's antecedents and consequent, and to activate the rules, i.e. we are not free to choose the shapes of their FOUs arbitrarily as we are, e.g., in most other engineering applications of IT2 fuzzy logic systems (e.g., [8]).

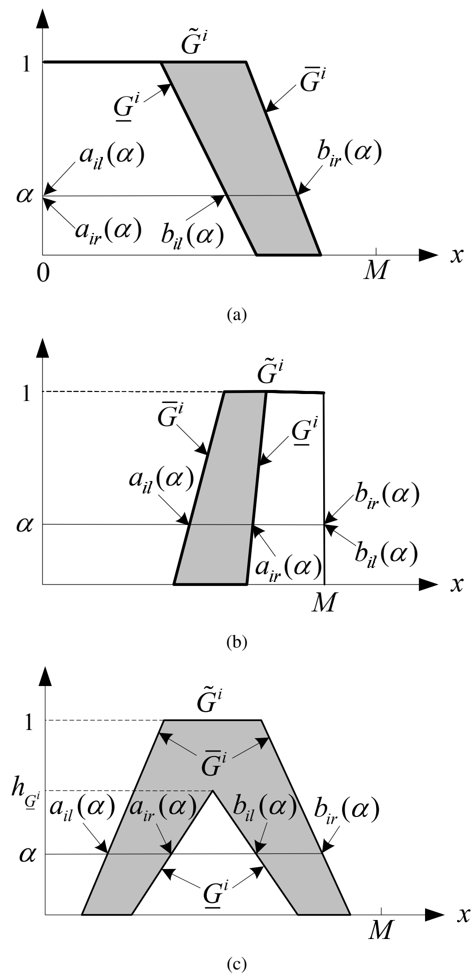

Fig. 1. Typical word FOUs. (a) left-shoulder, (b) right-shoulder, and (c) interior FOU.

Associated with (1) are the questions: How should one model the N rules? How should one model their inference mechanism? And, how should one combine multiple fired rules? These questions do not have unique answers, so, choices must be made. The following choices (i.e., assumptions) are made [10]:

Dongrui Wu and Jerry M. Mendel are with the Signal and Image Processing Institute, Ming Hsieh Department of Electrical Engineering, University of Southern California, 3740 McClintock Ave., Los Angeles, CA 90089-2564. Email: dongruiw@usc.edu; mendel@sipi.usc.edu.

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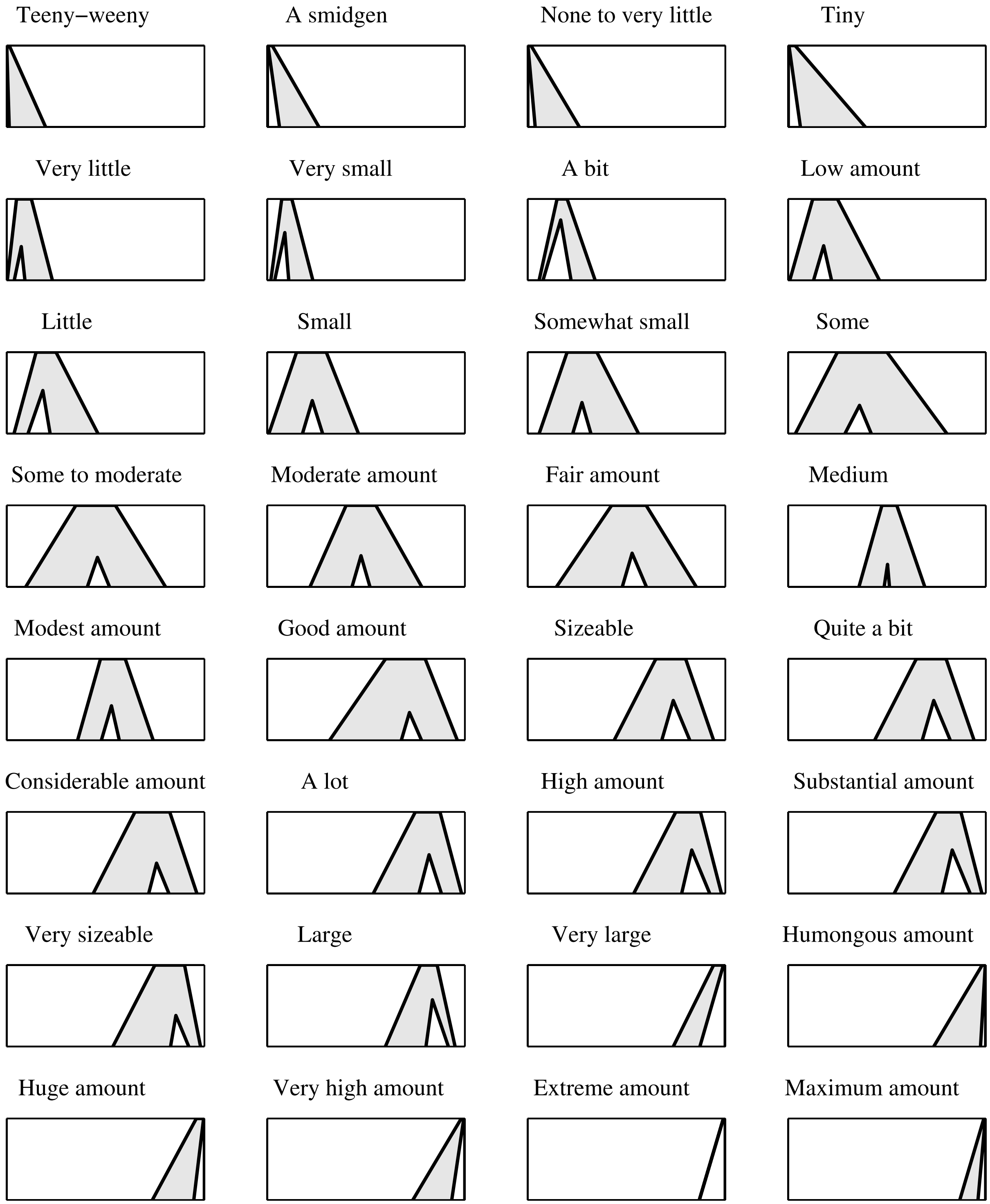

Fig. 2. The 32 word FOUs [6]. To read this figure, scan from left to right starting at the top of the page.

Assumptions: (1) The result of combining fired rules must lead to an FOU that resembles the three kinds of FOUs in a CWW codebook; (2) IT2 membership functions are separable; (3) No uncertainties are included about the connective word and in the rules; (4) Rules are activated by words that are modeled as either the shoulder or interior IT2 FSs that are depicted in Fig. 1; and, (5) Minimum t-norm is used for the *and* connective in rule antecedents. Of these five assumptions, Assumption 1 is the most challenging to achieve because it cannot be assumed a priori but must be demonstrated through analysis, and it is the one that we focus on.

Two fuzzy reasoning models that fit the concept of rational description are Mamdani and TSK, and both have been examined using IT2 FSs (e.g., [4], [8]); however, neither leads to a combined fired-rules output set that resembles the FOUs in the Fig. 1 codebook. Recall (e.g., see Fig. 5), that, even for type-1 FSs, each fired rule output FS for Mamdani implication that uses the minimum t-norm looks liked a clipped version of the consequent FS[1], and such a FS does not resemble the consequent FS. For a TSK model, the concept of a fired output FS does not occur, because its rule consequent is not a FS, but instead is a function of the inputs.

How fired rules are connected (combined) for a Mamdani model is open to interpretation. Zadeh connected rules [17] using the word *ELSE*, which is itself a bit vague. Some have interpreted the word *ELSE* as the *OR* connector, some have interpreted it as the *AND* connector, and not surprisingly, some have interpreted it as a blend of both the AND and OR connectors. Others prefer to perform the combining as a part of defuzzification. There is no measured evidence (data) to support any of these rule-combining methods for a Mamdani model when the objective is to make subjective judgments via approximate reasoning, as in CWW.

On the other hand, fired rules are easily combined using the TSK model through a weighted average of rule consequent functions, where the weights are the rule firing strengths. The result though is not a FS; it is a point value for type-1 FSs or an interval value for IT2 FSs. So, neither the Mamdani nor TSK models seem to be appropriate for CWW.

We have proposed [10] a new fuzzy reasoning model–Perceptual Reasoning (PR)–that not only fits the concept of rational description, but also satisfies Assumption 1 (as well as Assumptions 2-5), namely that *the result of combining fired rules must lead to an FOU that resembles the three kinds of FOUs in a CWW codebook.* PR consists of two steps:

1) A firing interval is computed for each rule, as would be done for both the IT2 Mamdani and TSK models, and
2) The IT2 FS consequents of the fired rules are combined using a special Linguistic Weighted Average (LWA) [12], [13] in which the weights are the firing intervals and the "signals" are the IT2 FS consequents.

Several properties of PR have been presented in [10]. However, because the LWA algorithms have been modified since then [15], in this paper we update the properties and also provide more properties for PR.

The rest of this paper is organized as follows: Section II introduces the algorithms of PR, Section III studies the properties of PR, and Section IV draws conclusions.

II. Perceptual Reasoning: Algorithms

A. Computing Firing Intervals

Because in PR both the antecedents and inputs are words modeled by IT2 FSs, the firing levels are intervals computed as [4], [8], [9]:

Theorem 1: Let the p inputs that activate a collection of M rules be denoted $\tilde{\mathbf{X}}'$. The result of the input and antecedent operations for the ith fired rule is contained in the firing interval $F^i(\tilde{\mathbf{X}}')$, where

$$F^i(\tilde{\mathbf{X}}') = [\underline{f}^i(\tilde{\mathbf{X}}'), \overline{f}^i(\tilde{\mathbf{X}}')] \equiv [\underline{f}^i, \overline{f}^i] \tag{3}$$

in which

$$\underline{f}^i(\tilde{\mathbf{X}}') = \sup_{\mathbf{x}} \int_{x_1\in X_1} \cdots \int_{x_p\in X_p} \left[\underline{X}_1(x_1) \star \underline{F}_1^i(x_1)\right] \star \cdots \star \left[\underline{X}_p(x_p) \star \underline{F}_p^i(x_p)\right] \Big/ \mathbf{x} \tag{4}$$

$$\overline{f}^i(\tilde{\mathbf{X}}') = \sup_{\mathbf{x}} \int_{x_1\in X_1} \cdots \int_{x_p\in X_p} \left[\overline{X}_1(x_1) \star \overline{F}_1^i(x_1)\right] \star \cdots \star \left[\overline{X}_p(x_p) \star \overline{F}_p^i(x_p)\right] \Big/ \mathbf{x} \tag{5}$$

and $\star$ denotes a t-norm, $\overline{X}_j$ and $\overline{F}_j^i$ are the upper membership functions (UMFs) of $\tilde{X}_j$ and $\tilde{F}_j^i$, respectively, and $\underline{X}_j$ and $\underline{F}_j^i$ are the lower membership functions (LMFs) of $\tilde{X}_j$ and $\tilde{F}_j^i$, respectively. ■

Though both minimum and product t-norms can be used in computing the firing intervals, for CWW we prefer the minimum t-norm for its simplicity. The detailed computations of (4) and (5) for the FOUs in Fig. 1 are presented in [11] and are omitted here, because they are not needed in this paper.

B. Combining the Fired Rules Using the LWA

Knowing the firing intervals $F^i(\tilde{\mathbf{X}}')$, fired rules can be combined using a LWA, denoted $\tilde{Y}_{PR}$, where subscripts "PR" denote Perceptual Reasoning. $\tilde{Y}_{PR}$ can be written in the following expressive[2] way:

$$\tilde{Y}_{PR} = \frac{\sum_{i=1}^n F^i(\tilde{\mathbf{X}}')\tilde{G}^i}{\sum_{i=1}^n F^i(\tilde{\mathbf{X}}')} \tag{6}$$

In (6) $F^i(\tilde{\mathbf{X}}')$ are intervals of non-negative real numbers, $\tilde{G}^i$ are rule-consequent IT2 FSs, and $n \le N$ is the number of fired rules, i.e. the rules whose firing intervals do not equal $[0, 0]$. This LWA is a special case of the more general LWA [12], [13] in which both $\tilde{G}^i$ and $F^i(\tilde{\mathbf{X}}')$ are IT2 FSs.

In the rest of this section, we provide a brief and highly condensed explanation of how to compute $\tilde{Y}_{PR}$. The reader

[1]When it uses the product t-norm it looks like a scaled version of the consequent FS.

[2]We refer to (6) as "expressive" because it is not computed using multiplications, additions and divisions, as expressed by it. Instead, $\underline{Y}_{PR}$ and $\overline{Y}_{PR}$ are computed separately using α-cuts, as explained later.

who is not interested in this can go directly to Section III, because the algorithms will be available for download.

C. Overviews

Algorithms for computing $\underline{Y}_{PR}$ and $\overline{Y}_{PR}$ are stated in Subsection D. Here we provide the definitions of the symbols used there.

In order to use the results in [12], [13], $F^i(\tilde{\mathbf{X}}')$ is interpreted here as an IT2 FS whose membership function is depicted in Fig. 3. Observe, in Fig. 3, each α-cut on $F^i(\tilde{\mathbf{X}}')$ is the *same* interval $[\underline{f}^i, \overline{f}^i]$, for $\forall\alpha \in [0,1]$, and that $\underline{F}^i(\tilde{\mathbf{X}}') = \overline{F}^i(\tilde{\mathbf{X}}')$.

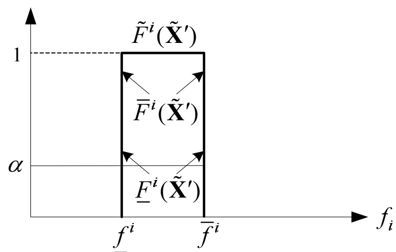

Fig. 3. $\tilde{F}^i(\tilde{\mathbf{X}}')$, the interpreted IT2 FS for firing interval $F^i(\tilde{\mathbf{X}}')$ of R^i.

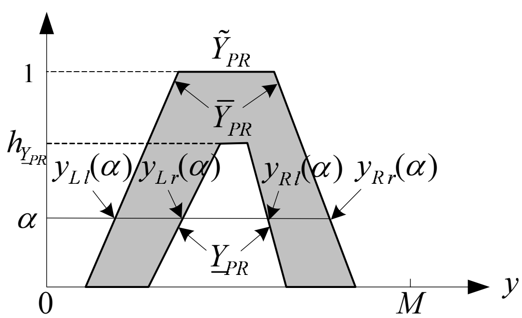

Fig. 4. $\tilde{Y}_{PR}$, the output of PR.

An interior FOU for $\tilde{G}^i$ is depicted in Fig. 1(c), in which the height of $\underline{G}^i$ is denoted $h_{\underline{G}^i}$, the α-cut on $\underline{G}^i$ is denoted[3] $[a_{ir}(\alpha), b_{il}(\alpha)]$, $\alpha \in [0, h_{\underline{G}^i}]$, and the α-cut on $\overline{G}^i$ is denoted $[a_{il}(\alpha), b_{ir}(\alpha)]$, $\alpha \in [0,1]$.

An interior FOU for $\tilde{Y}_{PR}$ is depicted in Fig. 4. The α-cut on $\overline{Y}_{PR}$ is $[y_{Ll}(\alpha), y_{Rr}(\alpha)]$ and the α-cut on $\underline{Y}_{PR}$ is $[y_{Lr}(\alpha), y_{Rl}(\alpha)]$, where, as explained in [12], [13], the end-points of these α-cut are computed as solutions to the following four optimization problems[4]:

$$y_{Ll}(\alpha) = \min_{\forall f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n a_{il}(\alpha) f_i}{\sum_{i=1}^n f_i}, \quad \alpha \in [0,1] \tag{7}$$

$$y_{Rr}(\alpha) = \max_{\forall f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n b_{ir}(\alpha) f_i}{\sum_{i=1}^n f_i}, \quad \alpha \in [0,1] \tag{8}$$

$$y_{Lr}(\alpha) = \min_{\forall f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n a_{ir}(\alpha) f_i}{\sum_{i=1}^n f_i}, \quad \alpha \in [0, h_{\underline{Y}_{PR}}] \tag{9}$$

$$y_{Rl}(\alpha) = \max_{\forall f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n b_{il}(\alpha) f_i}{\sum_{i=1}^n f_i}, \quad \alpha \in [0, h_{\underline{Y}_{PR}}] \tag{10}$$

where

$$h_{\underline{Y}_{PR}} = \min_i h_{\underline{G}^i} \tag{11}$$

Observe from (7) and (8) that $\overline{Y}_{PR}$, characterized by $[y_{Ll}(\alpha), y_{Rr}(\alpha)]$, is completely determined by $\overline{G}^i$, and from (9) and (10) that $\underline{Y}_{PR}$, characterized by $[y_{Lr}(\alpha), y_{Rl}(\alpha)]$, is completely determined by $\underline{G}^i$. Observe also from (7) and (8) that $\tilde{Y}_{PR}$ is always normal, i.e., its $\alpha = 1$ α-cut can always be computed. This is different from many other Approximate Reasoning methods, e.g., the Mamdani-inference based method. For the latter, even if only one rule is fired, unless the firing interval is $[1,1]$, the output is a clipped or scaled version of the original IT2 FS instead of a normal IT2 FS, as shown in Fig. 5. This may cause problems when the output is mapped to a word in the codebook.

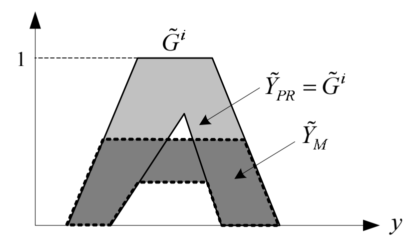

Fig. 5. The outputs of PR ($\tilde{Y}_{PR}$, the solid curve) and a Mamdani inference mechanism ($\tilde{Y}_M$, the dotted curve) when only Rule R^i is fired with firing interval $[0.3, 0.5]$.

D. Algorithms

In summary, knowing the firing intervals $[\underline{f}^i, \overline{f}^i]$, $i = 1, ..., n$, $\overline{Y}_{PR}$ is computed in the following way:

1) Calculate $y_{Ll}(\alpha_j)$ and $y_{Rr}(\alpha_j)$, $j = 1, ..., m$. To do this:
   a) Select appropriate m α-cuts for $\overline{Y}_{PR}$ (e.g., divide [0, 1] into $m-1$ intervals and set $\alpha_j = (j-1)/(m-1)$, $j = 1, 2, ..., m$).

[3] In this notation, the first subscript is an index that runs from 1 to at most n, whereas the second subscript is a pneumonic for left or right.

[4] The LWA used in this paper is slightly different from the version proposed in [12], [13] in that here $h_{\underline{Y}_{PR}} = \min_i h_{\underline{G}^i}$ whereas in [12], [13] $h_{\underline{Y}_{PR}}$ may be larger than $\min_i h_{\underline{G}^i}$. A detailed explanation about this is given in [15]. We advise the reader to use the LWA in this paper because it handles the case when $h_{\underline{G}^i}$ are not all the same correctly.

b) Find the α_j α-cut on $\overline{G}^i$ $(i = 1, ..., n)$; denote the end-points of its interval as $[a_{il}(\alpha_j), b_{ir}(\alpha_j)]$, respectively.
c) Use KM algorithms [2], [14] to find $y_{Ll}(\alpha_j)$ in (7) and $y_{Rr}(\alpha_j)$ in (8).
d) Repeat Steps (b)-(c) for every α_j $(j = 1, ..., m)$.

2) Construct $\overline{Y}_{PR}$ from the m α-cuts. To do this:
a) Store the left-coordinates $(y_{Ll}(\alpha_j), \alpha_j), j = 1, ..., m$.
b) Store the right-coordinates $(y_{Rr}(\alpha_j), \alpha_j), j = 1, ..., m$.
c) (Optional) Fit a spline curve through the $2m$ coordinates just stored.

Similarly, to compute $\underline{Y}_{PR}$:

1) Calculate $y_{Lr}(\alpha_j)$ and $y_{Rl}(\alpha_j)$, $j = 1, ..., m'$, where $\alpha_{m'} \leq \min_i h_{\underline{G}^i} \leq \alpha_{m'+1}$. To do this:
a) Find the α_j α-cut on $\underline{G}^i$ $(i = 1, ..., n)$.
b) Use KM algorithms [2], [14] to find $y_{Lr}(\alpha_j)$ in (9) and $y_{Rl}(\alpha_j)$ in (10).
c) Repeat Steps (a)–(b) for every α_j $(j = 1, \ldots, m')$.

2) Construct $\underline{Y}_{PR}$ from the m' α-cuts. To do this:
a) Store the left-coordinates $(y_{Lr}(\alpha_j), \alpha_j), j = 1, ..., m'$.
b) Store the right-coordinates $(y_{Rl}(\alpha_j), \alpha_j), j = 1, ..., m'$.
c) (Optional) Fit a spline curve through the $2m'$ coordinates just stored.

III. Perceptual Reasoning: Properties

Some properties of PR have been given in [10]. Because the LWA algorithms, which are the backbone of PR, have been modified [15], these properties have to be updated accordingly. The updated properties, as well as several new properties, are given in this section. All of these properties help demonstrate Assumption 1 for PR, namely, the result of combining fired rules using PR leads to an IT2 FS that resembles the three kinds of FOUs in a CWW codebook.

A. General Properties

Lemma 1: Let $y_{Lr}(\alpha)$ be defined in (9), where $a_{ir}(\alpha)$ have been sorted in ascending order and $\underline{f}^i \geq 0$. The properties of $y_{Lr}(\alpha)$ include [5]:

1) Because $y_{Lr}(\alpha)$ is a weighted average of $a_{ir}(\alpha)$, and $f^i \geq 0$,
$$a_{1r}(\alpha) \leq y_{Lr}(\alpha) \leq a_{nr}(\alpha). \tag{12}$$
2) $y_{Lr}(\alpha)$ is a non-decreasing function of $a_{ir}(\alpha)$.
3) $y_{Lr}(\alpha)$ can be re-expressed as
$$y_{Lr}(\alpha) = \min_{k \in [1, n-1]} \frac{\sum_{i=1}^k a_{ir}(\alpha)\overline{f}^i + \sum_{i=k+1}^n a_{ir}(\alpha)\underline{f}^i}{\sum_{i=1}^k \overline{f}^i + \sum_{i=k+1}^n \underline{f}^i} \tag{13}$$
and it can be computed by a KM algorithm or an enhanced KM algorithm [2], [8], [14]. ■

Note that $y_{Ll}(\alpha)$, $y_{Rl}(\alpha)$ and $y_{Rr}(\alpha)$ have similar properties. These properties will be used heavily in proving the theorems in this section.

Theorem 2: When all fired rules have the same consequent $\tilde{G}$, $\tilde{Y}_{PR}$ defined in (6) is the same as $\tilde{G}$. ■

An example where only one rule is fired is shown in Fig. 5.

Proof: When all fired rules have the same consequent $\tilde{G}$, (6) is simplified to[5]
$$\tilde{Y}_{PR} = \frac{\sum_{i=1}^n F^i(\tilde{\mathbf{X}}')\tilde{G}}{\sum_{i=1}^n F^i(\tilde{\mathbf{X}}')} \tag{14}$$
Denote the α-cut on $\overline{G}$ as $[a_l(\alpha), b_r(\alpha)]$ $(\alpha \in [0, 1])$ and the α-cut on $\underline{G}$ as $[a_r(\alpha), b_l(\alpha)]$ $(\alpha \in [0, h_{\underline{G}}])$. Then, the α-cuts on $\tilde{Y}_{PR}$, in (7)-(10), are computed as
$$y_{Ll}(\alpha) = \min_{f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n a_l(\alpha) f_i}{\sum_{i=1}^n f_i} = a_l(\alpha), \;\; \alpha \in [0, 1] \tag{15}$$
$$y_{Rr}(\alpha) = \max_{f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n b_r(\alpha) f_i}{\sum_{i=1}^n f_i} = b_r(\alpha), \;\; \alpha \in [0, 1] \tag{16}$$
$$y_{Lr}(\alpha) = \min_{f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n a_r(\alpha) f_i}{\sum_{i=1}^n f_i} = a_r(\alpha), \;\; \alpha \in [0, h_{\underline{G}}] \tag{17}$$
$$y_{Rl}(\alpha) = \max_{f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n b_{il}(\alpha) f_i}{\sum_{i=1}^n f_i} = b_l(\alpha), \;\; \alpha \in [0, h_{\underline{G}}] \tag{18}$$
i.e.,
$$[y_{Ll}(\alpha), y_{Rr}(\alpha)] = [a_l(\alpha), b_r(\alpha)], \quad \alpha \in [0, 1] \tag{19}$$
$$[y_{Lr}(\alpha), y_{Rl}(\alpha)] = [a_r(\alpha), b_l(\alpha)], \quad \alpha \in [0, h_{\underline{G}}] \tag{20}$$
Because every α-cut on $\tilde{Y}_{PR}$ is the same as the corresponding α-cut on $\tilde{G}$, it follows that $\tilde{Y}_{PR} = \tilde{G}$. ■

Theorem 3: $\tilde{Y}_{PR}$ is constrained by the consequents of the fired rules, i.e.,
$$\min_i a_{il}(\alpha) \leq y_{Ll}(\alpha) \leq \max_i a_{il}(\alpha) \tag{21}$$
$$\min_i a_{ir}(\alpha) \leq y_{Lr}(\alpha) \leq \max_i a_{ir}(\alpha) \tag{22}$$
$$\min_i b_{il}(\alpha) \leq y_{Rl}(\alpha) \leq \max_i b_{il}(\alpha) \tag{23}$$
$$\min_i b_{ir}(\alpha) \leq y_{Rr}(\alpha) \leq \max_i b_{ir}(\alpha) \tag{24}$$
The equalities hold simultaneously if and only if all fired rules have the same consequent. ■

Theorem 3 may be understood in this way: For PR using IT2 FSs, $\tilde{Y}_{PR}$ cannot be smaller than the smallest consequent of the fired rules, and it also cannot be larger than the largest consequent of the fired rules. A graphical illustration of Theorem 3 is shown in Fig. 6. Assume only two rules are fired and $\tilde{G}^1$ lies to the left of $\tilde{G}^2$; then, $\tilde{Y}_{PR}$ lies between $\tilde{G}^1$ and $\tilde{G}^2$.

Proof: (22) is readily seen from Part (1) of Lemma 1. The other three inequalities can be proved similarly.

[5] Recall that (14) is an "expressive" equation, so we cannot "cancel" $\sum_{i=1}^n F^i(\tilde{\mathbf{X}}')$ in its numerator and denominator.

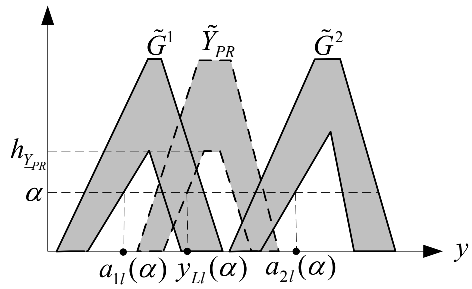

Fig. 6. A graphical illustration of Theorem 3, when only two rules fire.

When all n fired rules have the same consequent $\tilde{G}$, we know from Theorem 2 that

$$y_{Ll}(\alpha) = \min_{\forall i} a_{il}(\alpha) = \max_{\forall i} a_{il}(\alpha) = a_l(\alpha) \quad (25)$$

$$y_{Lr}(\alpha) = \min_{\forall i} a_{ir}(\alpha) = \max_{\forall i} a_{ir}(\alpha) = a_r(\alpha) \quad (26)$$

$$y_{Rl}(\alpha) = \min_{\forall i} b_{il}(\alpha) = \max_{\forall i} b_{il}(\alpha) = b_l(\alpha) \quad (27)$$

$$y_{Rr}(\alpha) = \min_{\forall i} b_{ir}(\alpha) = \max_{\forall i} b_{ir}(\alpha) = b_r(\alpha) \quad (28)$$

When all $\tilde{G}^i$ are not the same, at least one of (25)-(28) does not hold. Hence, equalities in (21)-(24) hold simultaneously if and only if all fired rules have the same consequent. ■

Theorem 4: Generally, $\underline{Y}_{PR}$ is trapezoidal-looking[6]; however, $\underline{Y}_{PR}$ is triangle-looking[7] when all $\underline{G}_i$ are triangles with the same height h, and either of the following unlikely events is true:

1) the apexes of all $\underline{G}^i$ coincide, or
2) $\underline{f}^i = \overline{f}^i$ for $\forall i$. ■

Proof: Because (see Fig. 1) $b_{il}(\alpha) \geq a_{ir}(\alpha)$, we see from (9) and (10) that

$$\begin{aligned} y_{Lr}(\alpha) &= \min_{\forall f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n a_{ir}(\alpha) f_i}{\sum_{i=1}^n f_i} \\ &\leq \max_{\forall f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n a_{ir}(\alpha) f_i}{\sum_{i=1}^n f_i} \\ &\leq \max_{\forall f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n b_{il}(\alpha) f_i}{\sum_{i=1}^n f_i} \\ &= y_{Rl}(\alpha) \end{aligned} \quad (29)$$

i.e., $y_{Lr}(\alpha) \leq y_{Rl}(\alpha)$, so that in general $\underline{Y}_{PR}$ is trapezoidal-looking.

When all $\underline{G}_i$ are triangles with the same height h, according to (11), $h_{\underline{Y}_{PR}} = \min_i h_{\underline{G}^i} = h$. When the apexes of all $\underline{G}^i$ coincide at $x = \lambda$, the $\alpha = h$ α-cuts on them collapse to a point, i.e., $a_{ir}(h) = b_{il}(h) = \lambda$ for $\forall i = 1, \ldots, n$. Consequently,

$$\begin{aligned} y_{Lr}(h) &= \min_{f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n a_{ir}(h) f_i}{\sum_{i=1}^n f_i} \\ &= \lambda \min_{f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n f_i}{\sum_{i=1}^n f_i} = \lambda \end{aligned} \quad (30)$$

$$\begin{aligned} y_{Rl}(h) &= \max_{f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n b_{il}(h) f_i}{\sum_{i=1}^n f_i} \\ &= \lambda \max_{f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n f_i}{\sum_{i=1}^n f_i} = \lambda \end{aligned} \quad (31)$$

i.e., $y_{Lr}(h) = y_{Rl}(h) = \lambda$; hence, $\underline{Y}_{PR}$ is triangle-looking with height h.

When all $\underline{G}_i$ are triangles with the same height h, $a_{ir}(h) = b_{il}(h)$, and $\underline{Y}_{PR}$ also has height h. If $\underline{f}^i = \overline{f}^i \equiv f_i$,

$$\begin{aligned} y_{Lr}(h) &= \min_{f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n a_{ir}(h) f_i}{\sum_{i=1}^n f_i} \\ &= \frac{\sum_{i=1}^n a_{ir}(h) f_i}{\sum_{i=1}^n f_i} \end{aligned} \quad (32)$$

$$\begin{aligned} y_{Rl}(h) &= \max_{f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n b_{il}(h) f_i}{\sum_{i=1}^n f_i} \\ &= \frac{\sum_{i=1}^n b_{il}(h) f_i}{\sum_{i=1}^n f_i} \\ &= \frac{\sum_{i=1}^n a_{ir}(h) f_i}{\sum_{i=1}^n f_i} \\ &= y_{Lr}(h) \end{aligned} \quad (33)$$

hence, $\underline{Y}_{PR}$ is again triangle-looking with height h. ■

Theorem 5: Generally, $\overline{Y}_{PR}$ is trapezoidal-looking; however, $\overline{Y}_{PR}$ is triangle-looking when either of the following unlikely events is true:

1) all $\overline{G}^i$ are triangles and their apexes coincide, or
2) $\underline{f}^i = \overline{f}^i$ for $\forall i$. ■

Proof: Because all $\overline{G}^i$ have equal height 1, the approach of the proof of Theorem 4 can be used here. ■

B. Properties Related to Assumption 1

In this subsection we show that $\tilde{Y}_{PR}$ computed from (6) resembles the three kinds of FOUs in a CWW codebook. The following three lemmas are used in the proofs of Theorems 6-8:

Lemma 2: An IT2 FS $\tilde{Y}_{PR}$ is a left shoulder [see Fig. 7(a)] if and only if $y_{Ll}(1) = 0$ and $y_{Lr}(h_{\underline{Y}_{PR}}) = 0$. ■

Proof: Intuitively, an IT2 FS $\tilde{Y}_{PR}$ is a left shoulder (see Fig. 4) if and only if $y_{Ll}(\alpha) = 0$ for $\forall \alpha \in [0, 1]$ and $y_{Lr}(\alpha) = 0$ for $\forall \alpha \in [0, h_{\underline{Y}_{PR}}]$, as shown in Fig. 7(a). Because only convex IT2 FSs are used in PR, we have $y_{Ll}(\alpha) \leq y_{Ll}(1)$ for $\forall \alpha \in [0, 1]$. Consequently, $y_{Ll}(1) = 0$ means $y_{Ll}(\alpha) = 0$ for $\forall \alpha \in [0, 1]$. Similarly, $y_{Lr}(h_{\underline{Y}_{PR}}) = 0$ means $y_{Lr}(\alpha) = 0$ for $\forall \alpha \in [0, h_{\underline{Y}_{PR}}]$. ■

Lemma 3: An IT2 FS $\tilde{Y}_{PR}$ is a right shoulder [see Fig. 7(b)] if and only if $y_{Rr}(1) = M$ and $y_{Rl}(h_{\underline{Y}_{PR}}) = M$. ■

[6] $\underline{Y}_{PR}$ is trapezoidal-looking if its $\alpha = h_{\underline{Y}_{PR}}$ α-cut is an interval instead of a single point, e.g., $\underline{Y}_{PR}$ in Fig. 4 is trapezoidal-looking.

[7] $\underline{Y}_{PR}$ is triangle-looking if its $\alpha = h_{\underline{Y}_{PR}}$ α-cut converges to a single point, e.g., $\underline{Y}_{PR}$ in Fig. 5 is triangle-looking.

The proof is similar to that of Lemma 2, and hence is left to the reader.

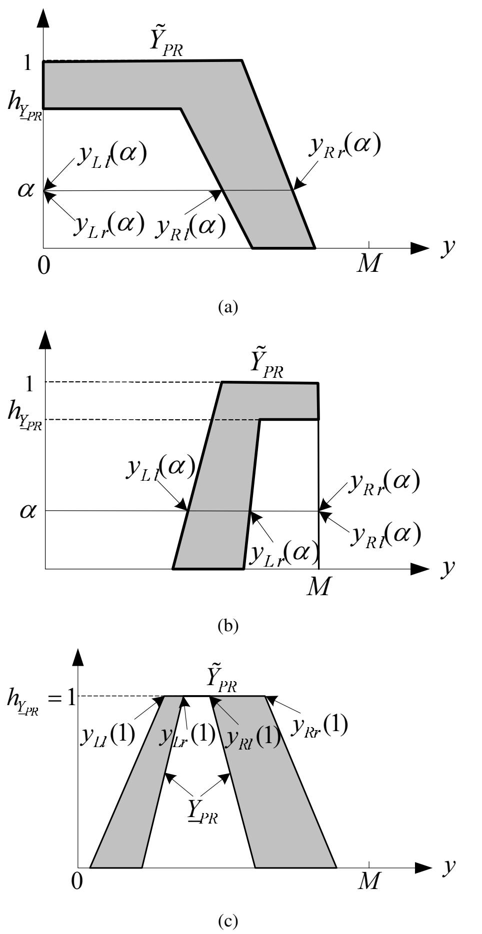

Fig. 7. α-cuts on a (a) left shoulder, (b) right shoulder, and (c) interior $\tilde{Y}_{PR}$ all with $h_{\underline{Y}_{PR}} = 1$.

Lemma 4: An IT2 FS $\tilde{Y}_{PR}$ is an interior FOU if and only if $y_{Lr}(h_{\underline{Y}_{PR}}) > 0$ and $y_{Rl}(h_{\underline{Y}_{PR}}) < M$. ■

Proof: When $y_{Lr}(h_{\underline{Y}_{PR}}) > 0$ and $y_{Rl}(h_{\underline{Y}_{PR}}) < M$, $\tilde{Y}_{PR}$ is not a left shoulder by Lemma 2, and it is also not a right shoulder by Lemma 3. Consequently, $\tilde{Y}_{PR}$ must be an interior FOU. ■

Examples of interior $\tilde{Y}_{PR}$ are shown in Figs. 4 and 7(c).

The main properties of $\tilde{Y}_{PR}$ are given next as a collection of three theorems.

Theorem 6: Let $\tilde{Y}_{PR}$ be defined in (6). Then, $\tilde{Y}_{PR}$ is a left shoulder if and only if:

1) At least one $\tilde{G}^i$ is a left shoulder; and,
2) For every $\tilde{G}^i$ which is not a left shoulder, the corresponding firing interval satisfies $\underline{f}^i = 0$. ■

Comment: Theorem 6 demonstrates that $\tilde{Y}_{PR}$ is a left shoulder does not necessarily mean all consequents of the fired rules must be left shoulders.

Proof: Recall that $y_{Ll}(1)$ is computed as

$$y_{Ll}(1) = \min_{\forall f_i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^n a_{il}(1) f_i}{\sum_{i=1}^n f_i} \tag{34}$$

We consider two cases for (34):

1) All $\tilde{G}^i$ are left shoulders, i.e., all $a_{il}(1) = 0$. Obviously, $y_{Ll}(1) = 0$ in this case. Similarly, we can show that $y_{Lr}(h_{\underline{Y}_{PR}}) = 0$; hence, according to Lemma 2, $\tilde{Y}_{PR}$ is a left shoulder.
2) Not all $\tilde{G}^i$ are left shoulders, i.e., not all $a_{il}(1)$ are 0. In this case $y_{Ll}(1)$ needs to be computed using a KM algorithm, and $\{a_{il}(1)\}$ need to be sorted in ascending order first. Assume K ($1 \le K < n$) of the n $\tilde{G}^i$ are left shoulders. Because left shoulders have $a_{il}(1) = 0$, in the sorted $\{a_{il}(1)\}$

$$a_{il}(1) \begin{cases} = 0, & i = 1, \ldots, K \\ > 0, & i = K+1, \ldots, n \end{cases} \tag{35}$$

According to Part 1 of Lemma 1,

$$y_{Ll}(1) \ge a_{1l}(1) = 0 \tag{36}$$

According to Part 3 of Lemma 1, and also using the assumed fact that $\underline{f}^i = 0$ for $\forall i \ge K+1$,

$$\begin{aligned} y_{Ll}(1) &= \min_{k\in[1,n-1]} \frac{\sum_{i=1}^k a_{il}(1)\overline{f}^i + \sum_{i=k+1}^n a_{il}(1)\underline{f}^i}{\sum_{i=1}^k \overline{f}^i + \sum_{i=k+1}^n \underline{f}^i} \\ &\le \frac{\sum_{i=1}^K a_{il}(1)\overline{f}^i + \sum_{i=K+1}^n a_{il}(1)\underline{f}^i}{\sum_{i=1}^K \overline{f}^i + \sum_{i=K+1}^n \underline{f}^i} \\ &= 0 \end{aligned} \tag{37}$$

(36) and (37) together demonstrate that $y_{Ll}(1) = 0$. Similarly, we can show that $y_{Lr}(h_{\underline{Y}_{PR}}) = 0$. From Lemma 2 we know $\tilde{Y}_{PR}$ is a left-shoulder.

In summary, $\tilde{Y}_{PR}$ is a left-shoulder only when:

Case 1: All $\tilde{G}^i$ are left shoulders; or

Case 2: At least one $\tilde{G}^i$ is a left shoulder, and the remaining $\tilde{G}^i$ have $\underline{f}^i = 0$.

Because Case 1 is included in Case 2, we only present Case 2 in the statement of Theorem 6. ■

Theorem 7: Let $\tilde{Y}_{PR}$ be defined in (6). Then, $\tilde{Y}_{PR}$ is a right shoulder if and only if:

1) At least one $\tilde{G}^i$ is a right shoulder; and,
2) For every $\tilde{G}^i$ which is not a right shoulder, the corresponding firing interval satisfies $\underline{f}^i = 0$. ■

The proof is similar to that of Theorem 6, and hence is omitted here.

Comment: Theorem 7 demonstrates that $\tilde{Y}_{PR}$ is a right shoulder does not necessarily mean all consequents of the fired rules must be right shoulders.

Theorem 8: Let $\tilde{Y}_{PR}$ be defined in (6). Then, $\tilde{Y}_{PR}$ is an interior FOU if $\tilde{G}^i$ do not satisfy the requirements in Theorems 5 and 6. More specifically, $\tilde{Y}_{PR}$ is an interior FOU if and only if:

1) All $\tilde{G}^i$ are interior FOUs; or,
2) $\tilde{G}^i$ consist of more than one kind of shapes, and for each of at least two kinds of shapes, there exists at least one corresponding firing interval such that $\underline{f}^i > 0$. ■

The correctness of Theorem 8 is readily seen from Theorems 6 and 7, i.e., when either of Cases 1 and 2 is true, $\tilde{Y}_{PR}$ is neither a left shoulder nor a right shoulder, and hence it must be an interior FOU.

Comment: Theorem 8 demonstrates that $\tilde{Y}_{PR}$ is an interior FOU does not necessarily mean all consequents of the fired rules must be interior FOUs.

Theorems 6-8 are important because they show that the output of PR is normal and similar to the word FOUs in a codebook (see Fig. 2). So, a similarity measure [16] can be used to map $\tilde{Y}_{PR}$ to a word in the codebook. On the other hand, it is less intuitive to map a clipped FOU (see $\tilde{Y}_M$ in Fig. 5), as obtained from a Mamdani inference mechanism, or a crisp point, as obtained from the TSK inference mechanism, to a normal word FOU in the codebook.

IV. CONCLUSIONS

In this paper, the algorithms and properties of PR (which is a CWW engine proposed in [10]), have been introduced. PR uses IF-THEN rules; however, unlike traditional IF-THEN rules, which are combined using Mamdani or TSK inference mechanisms, an LWA is used to combine the fired rules. The main advantage of PR is that its output FOU resembles the three types of input FOUs in a CWW codebook, left-shoulder, right-shoulder, or interior FOU. This is very different from Mamdani and TSK models, none of which have this property. In future works, we will apply PR to CWW applications and compare it with traditional Mamdani and TSK models.

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