Perceptual Reasoning: A New Computing With Words Engine

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***Abstract*— Zadeh proposed the paradigm of *computing with words* (CWW). We have proposed a CWW architecture for making subjective judgments, called a *Perceptual Computer* (Per-C). Because words mean different things to different people, the Per-C uses interval type-2 fuzzy sets (IT2 FSs). The *encoder* of the Per-C transforms words, in an application-dependent word-codebook, into IT2 FSs. The central element of the Per-C is the *CWW engine*, which maps IT2 FSs to IT2 FSs. Several CWW engines have appeared in the literature, e.g., fuzzy IF-THEN rules to perform inference and/or reasoning based on Mamdani or TSK models, linguistic weighted averages (LWAs) to aggregate linguistic data, and linguistic summarization to perform human friendly data mining. In this paper a new CWW engine–*Perceptual Reasoning* (PR)–is proposed. It also uses fuzzy IF-THEN rules; however, unlike a traditional Mamdani or TSK model, in which fired rules are combined using the union, or addition, or during the defuzzification process, in PR a LWA is used to combine the fired rules. We prove that the output IT2 FSs of PR can only look like the IT2 FSs in the application codebook. This is very important for CWW, because the last component of the Per-C is a *decoder* which converts the CWW output IT2 FS back into a word, e.g. a word whose IT2 FS is most similar to it.**

***Index Terms*— Computing with words, perceptual reasoning, perceptual computer, interval type-2 fuzzy sets, linguistic weighted average**

I. INTRODUCTION

Zadeh coined the phrase "*computing with words*" (CWW) [26], [27]. According to him, CWW is *"a methodology in which the objects of computation are words and propositions drawn from a natural language."* Nikravesh [17] pointed out that CWW is *"fundamentally different from the traditional expert systems which are simply tools to 'realize' an intelligent system, but are not able to process natural language which is imprecise, uncertain and partially true."*

Our thesis is that *words mean different things to different people* [10] and so there is uncertainty associated with words, which means that fuzzy logic must somehow use this uncertainty when it computes with words [9], [10]. Hence, we argue that interval type-2 fuzzy sets (IT2 FSs) should be used in CWW [12]. We will limit our discussions to IT2 FSs in this paper.

A specific architecture depicted in Fig. 1 is proposed in [14] and [11] for making (subjective) judgements by CWW. It is called a *Perceptual Computer*–Per-C for short, and its use is called *Perceptual Computing*. Perceptions (i.e., granulated terms, words) activate the Per-C and are also output by the Per-C; so, it is possible for a human to interact with the Per-C just using a vocabulary of words. In Fig. 1, the *encoder*[1] transforms linguistic perceptions into IT2 FSs that activate a *CWW engine*. How to do this most simply is explained in [8], and is not the subject of this paper, although some aspects of it are discussed below. The *decoder*[2] maps the output of the CWW engine back into a word. Usually a codebook is available, in which every word (the vocabulary) is modeled as an IT2 FS. The output of the CWW engine is mapped into a word (in that vocabulary) most similar to it. How to do this is explained in [24] and is also not the subject of this paper, although some aspects of this will also be explained below. The CWW engine, e.g. IF-THEN rules (e.g., [9]), the linguistic weighted average [22], [23], linguistic summarizations [5], [12], etc, maps IT2 FSs into IT2 FSs. In this paper, *we focus only on CWW engines that are rule-based and the computations that map its input IT2 FSs into its output IT2 FSs.*

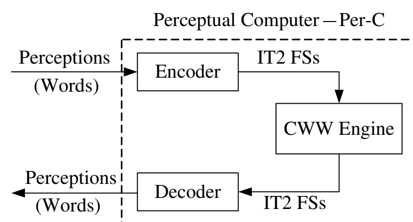

Fig. 1. Architecture of the Perceptual Computer.

In order to carry out those computations one must first ask: "What kinds of IT2 FSs can activate such a rule-based CWW engine?" The answer to this question has been obtained by Liu and Mendel [8], as explained next. In their encoding method, data intervals about a vocabulary of words are obtained[3] from a group of subjects. After some pre-processing, during which some intervals are eliminated (e.g., outliers), each of the remaining intervals is mapped into a triangular[4] type-1 (T1) membership function (MF). Then the union of all of these MFs is taken, after which the union is upper and lower bounded.

[1] Zadeh calls this *constraint explicitation* in [26], [27]. In [28] and some of his recent talks, he calls this *precisiation*.

[2] Zadeh calls this *linguistic approximation* in [26], [27].

[3] Subjects are asked "On a scale of 0-10 where would you locate the end-points of an interval that you associate with the word ____?"

[4] Mapping a uniformly distributed confidence interval into a triangular type-1 MF has been shown to be quite natural and mathematically sound in [3]. Although our intervals are not confidence intervals, there seems to be a connection between [3] and [8], one that we are currently exploring.

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The result is the footprint of uncertainty (FOU) for an IT2 FS[5], which is completely described by these lower and upper bounds, called the lower membership function (LMF) and the upper membership function (UMF), respectively.

Surprisingly, when this methodology was applied to real data [8] only three general kinds of FOUs emerged, namely left-shoulder FOUs, right-shoulder FOUs and interior FOUs (see Fig. 2). These FOUs have the following general features:

1) Left- and right-shoulder FOUs: The legs of the LMF and UMF are not parallel.
2) Interior FOUs: The UMF is a trapezoid that usually is not symmetrical, and the LMF is a triangle that usually is not symmetrical.

Note that these results are due to mapping each subjects interval into a triangular MF, and do not change even if the interval is mapped into another MF shape such as a trapezoidal or a Gaussian, when the upper and lower bounds are approximated using piece-wise linear functions [12].

So, in a rule-based CWW engine only a very limited number of IT2 FSs can activate the rules, and we are not free to choose the shapes of their FOUs arbitrarily, as we are, e.g. in most other engineering applications of interval type-2 fuzzy logic systems (e.g., [9]).

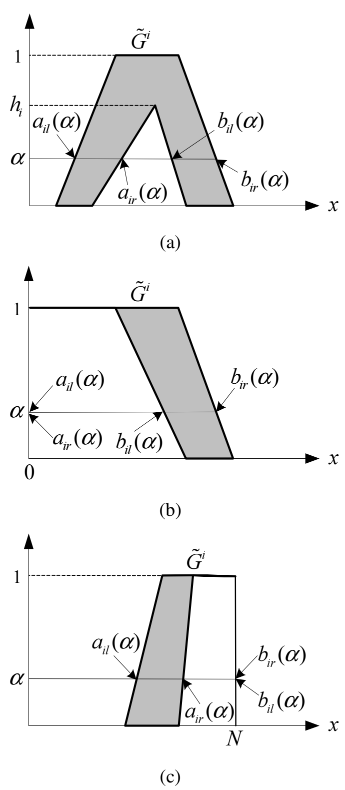

Fig. 2. FOUs. (a) interior, (b) left-shoulder, and (c) right-shoulder.

[5]It is assumed that readers are familiar with IT2 FSs. If they are not, see, e.g. [13].

The general structure of a rule-based CWW Engine is: There are p inputs $x_1 \in X_1$, $x_2 \in X_2$, ..., $x_p \in X_p$ and one output $y \in Y$, and M rules, each of the form:

$$R^i: \text{ If } x_1 \text{ is } \tilde{F}_1^i \text{ and } \ldots \text{ and } x_p \text{ is } \tilde{F}_p^i, \text{ Then } y \text{ is } \tilde{G}^i \quad (1)$$

In this rule the p antecedents and the consequent are modeled as IT2 FSs that are a subset of the words in a CWW codebook; hence, they can only be IT2 FSs like the ones shown in Fig. 2.

Comment: The codebook for a CWW application may be rather large, so that users who interface with the Per-C can operate in a user-friendly environment. Usually, only a small subset of the words in the codebook would be used to establish the M rules, especially when rules are extracted from experts. What is important is that the words used to characterize each of the p antecedents and the consequent lead to FOUs that cover the domain of each antecedent. In our experience, 3-7 words will cover an interval, e.g. 0-10. ■

How one should model the M rules, their inference mechanism and the combining of multiple fired rules for a Per-C that is used for perceptual computing are questions that do not have unique answers; so, choices must be made. In this paper, the following choices (i.e., assumptions) are made:

Assumptions: (1) The result of combining fired rules must lead to an FOU that resembles the three kinds of FOUs in a CWW codebook; (2) IT2 MFs are separable; (3) No uncertainties are included about connective words; (4) Rules are activated by words that are modeled as either shoulder or interior IT2 FSs; and, (5) Minimum *t*-norm is used for *and* connective in rule antecedents. Because these assumptions are so important, brief discussions are provided next for each of them.

1) *The result of combining fired rules must lead to an FOU that resembles the three kinds of FOUs in a CWW codebook:* This is a very plausible requirement, since the decoder in the Per-C maps the CWW output FOU into a word in the codebook most similar to it. We will have much more to say about this requirement in Section II.
2) *IT2 MFs are separable:* Because each word in the vocabulary has been modeled independently, separable MFs seem reasonable.
3) *No uncertainties are included about connective words:* Although there exists a literature (e.g., [19]–[21] and [25]) for allowing the connective words *and* and *or* to incorporate uncertainties, except for [25] all results are for T1 FS antecedents and consequents, and even those results are very complicated. [25]'s results are even more complicated. Our first approach to a CWW rule-based engine is to keep it as simple as possible, and to see if sensible results can be obtained. If they cannot be, then one possibility is to use more complicated models for connector words, but, they must be in the context of IT2 FS models for words.
4) *Rules are activated by words that are modeled as either shoulder or interior IT2 FSs:* Rules will be activated by words that are in the codebook, and as we have explained these words will be modeled as assumed above.

5) *Minimum t-norm is used for and connective:* In a rule-based fuzzy logic system (FLS) product and minimum t-norms are most popular. We have found that computing the sup-min composition in closed form is relatively straightforward, but computing the sup-product composition is very difficult [16]. Because the Per-C is very different from the more popular function approximation use for an IT2 FLS, in which universal approximation dominates and product t-norm is most popular, there is no compelling reason to use product t-norm over the minimum t-norm. So, we have taken the pragmatic approach and have chosen to focus on the minimum t-norm.

The rest of this paper is organized as follows: Section II introduces a kind of reasoning that satisfies Assumption 1, and which we call Perceptual Reasoning. Section III describes how to compute the firing intervals, Section IV describes how to combine the fired rules, and Section V draws conclusions.

II. Perceptual Reasoning

A. Introduction

There are many models for the fuzzy implication, under the rubric of approximate reasoning, e.g. Table 11.1 in [6] lists 14. Each of these models has the property that it reduces to the truth table of material implication when fuzziness disappears, and to-date none of these models has been examined using IT2 FSs. Following is a quote from [2] that we have found to be very illuminating:

> Rational calculation is the view that the mind works by carrying out probabilistic, logical, or decision-theoretic operations. ... Rational calculation is explicitly avowed by relatively few theorists, though it has clear advocates with respect to logical inference. Mental logicians propose that much of cognition is a matter of carrying out logical calculations (e.g., [1], [4], [18]) ... Rational description, by contrast, is the view that behavior can be approximately described as conforming with the results that would be obtained by some rational calculation. This view does not assume (though it does not rule out) that the thought processes underlying behavior involves any rational calculation.

For the Per-C we do not implement logical reasoning as prescribed by the truth table of material implication; instead we subscribe to rational description.

Two fuzzy reasoning models that fit the concept of rational description are Mamdani and TSK, and both have been examined using IT2 FSs (e.g., [7], [9]); however, neither leads to a combined fired-rules output set that resembles the FOUs in our codebook (Fig. 2). Recall (e.g., see Fig. 6 in [13]), that even for T1 FSs each fired rule output FS for Mamdani implication that uses, e.g. the minimum t-norm looks liked a clipped version of the consequent FS[6], and such a FS does not resemble the consequent FS. For a TSK model, the concept of a fired output FS does not occur, because the rule consequent in a TSK rule is not a FS, but is a function of the inputs.

How fired rules are connected (combined) for a Mamdani model is open to interpretation. Zadeh connected rules [29] using the word *ELSE*, which is itself a bit vague. Some have interpreted the word *ELSE* as the *OR* connector, some have interpreted it as the *AND* connector, and not surprisingly, some have interpreted it as a blend of both the AND and OR connectors. Others prefer to perform the combining as a part of defuzzification. There is no measured evidence (data) to support any of these rule-combining methods for a Mamdani model when the objective is to make subjective judgments.

Interestingly enough, fired rules are easily combined using the TSK model through a weighted average of rule consequent functions, where the weights are the rule firing strengths. The result though is not a FS; it is a point value for T1 FSs or an interval value for IT2 FSs. So, neither the Mamdani nor TSK models seem to be appropriate for the Per-C.

B. Perceptual Reasoning Described

We now propose a new fuzzy reasoning model–Perceptual Reasoning–that not only fits the concept of rational description, but also satisfies Assumption 1, namely that *the result of combining fired rules must lead to an FOU that resembles the three kinds of FOUs in a CWW codebook.*

Perceptual Reasoning consists of two steps:

1) A firing interval is computed for each rule, as would be done for both the IT2 FS Mamdani and TSK models, and
2) The IT2 FS consequents of the fired rules are combined using a Linguistic Weighted Average (LWA) [22], [23] in which the weights are the firing intervals and the "signals" are the IT2 FS consequents.

Firing interval calculations are covered in Section III, and aggregation of the IT2 FS consequents using the LWA is covered in Section IV. In Section IV we also prove that the output of the LWA is an IT2 FS whose FOU resembles the three kinds of word FOUs in our codebook, i.e. it looks like the ones in Fig. 2.

III. Computing Firing Intervals

In the IT2 FLS literature (e.g., [6], [9], [15]) computing the firing interval is simplest when inputs are modeled as singletons, more difficult when inputs are modeled as T1 FSs, and most difficult when inputs are modeled as IT2 FSs. Because our rules are always activated by IT2 FSs we must immediately be concerned with computing the firing interval for this most difficult case. Following is the computation of the firing interval for this case [7], [9], [15]:

Theorem 1: Let the p inputs that activate a collection of M rules be denoted $\tilde{\mathbf{X}}'$. The result of the input and antecedent operations for the ith fired rule is contained in the firing interval $F^i(\tilde{\mathbf{X}}')$, where

$$F^i(\tilde{\mathbf{X}}') = [\underline{f}^i(\tilde{\mathbf{X}}'), \overline{f}^i(\tilde{\mathbf{X}}')] \equiv [\underline{f}^i, \overline{f}^i] \quad (2)$$

[6]When it uses the product t-norm it looks like a scaled version of the consequent FS.

in which

$$\underline{f}^i(\tilde{\mathbf{X}}') = \sup_{\mathbf{x}} \int_{x_1 \in X_1} \cdots \int_{x_p \in X_p} \left[\underline{\mu}_{\tilde{X}_1}(x_1) \star \underline{\mu}_{\tilde{F}_1^i}(x_1)\right] \star \cdots \star \left[\underline{\mu}_{\tilde{X}_p}(x_p) \star \underline{\mu}_{\tilde{F}_p^i}(x_p)\right] \Big/ \mathbf{x} \quad (3)$$

$$\overline{f}^i(\tilde{\mathbf{X}}') = \sup_{\mathbf{x}} \int_{x_1 \in X_1} \cdots \int_{x_p \in X_p} \left[\overline{\mu}_{\tilde{X}_1}(x_1) \star \overline{\mu}_{\tilde{F}_1^i}(x_1)\right] \star \cdots \star \left[\overline{\mu}_{\tilde{X}_p}(x_p) \star \overline{\mu}_{\tilde{F}_p^i}(x_p)\right] \Big/ \mathbf{x} \quad (4)$$

and $\star$ denotes a t-norm. ■

Though both minimum and product t-norms can be used in computing the firing intervals, we prefer the minimum t-norm for its simplicity. The detailed computations of (3) and (4) are presented in [16] and are omitted in this paper.

IV. Combining the Fired Rules Using the LWA

In this section it is assume that, for a given input $\tilde{\mathbf{X}}'$, firing levels $F^i(\tilde{\mathbf{X}}')$ [see (2)] have been computed for all fired rules. The LWA for Perceptual Reasoning, $\tilde{Y}_{PR}$, can be written in the following expressive[7] way:

$$\tilde{Y}_{PR} = \frac{\sum_{i=1}^m F^i(\tilde{\mathbf{X}}')\tilde{G}^i}{\sum_{i=1}^m F^i(\tilde{\mathbf{X}}')} \quad (5)$$

In (5) $F^i(\tilde{\mathbf{X}}')$ are intervals of non-negative real numbers, $\tilde{G}^i$ are IT2 FSs, and $m \le M$ is the number of fired rules, i.e. the rules whose firing intervals do not equal $[0, 0]$. This LWA is a special case of the more general LWA in which both $\tilde{G}^i$ and $F^i(\tilde{\mathbf{X}}')$ are IT2 FSs.

$\tilde{Y}_{PR}$ is an IT2 FS and is therefore completely described by its lower and upper MFs, $LMF(\tilde{Y}_{PR})$and $UMF(\tilde{Y}_{PR})$, respectively. How to compute $LMF(\tilde{Y}_{PR})$and $UMF(\tilde{Y}_{PR})$ using α-cuts is explained in [22], [23], and although all of the details of these calculations are unnecessary for the present paper, certain results are needed, and those are the ones focused on next.

Because [22], [23] is for a more general LWA in which both $\tilde{G}^i$ and F^i are IT2 FSs, and in the present case F^i is not an IT2 FS, in order to use the results in [22], [23] F^i is interpreted here as a T1 FS whose MF is depicted in Fig. 3. Note that this T1 FS can in turn be interpreted as an FOU in which $LMF(\tilde{F}^i(\tilde{\mathbf{X}}')) = UMF(\tilde{F}^i(\tilde{\mathbf{X}}'))$, so that every point in the interval $[\underline{f}^i, \overline{f}^i]$ has membership $[1, 1]$. Observe, in Fig. 3, each α-cut on $F^i(\tilde{\mathbf{X}}')$ is the *same* interval[8] $[\underline{f}^i, \overline{f}^i]$, for $\forall\alpha \in [0, 1]$.

An interior FOU for $\tilde{G}^i$ is depicted in Fig. 2(a), in which the height of $LMF(\tilde{G}^i)$ is denoted h_i, the α-cut on $LMF(\tilde{G}^i)$ is denoted[9] $[a_{ir}(\alpha), b_{il}(\alpha)]$ ($\alpha \in [0, h_i]$), and the α-cut on $UMF(\tilde{G}^i)$is denoted $[a_{il}(\alpha), b_{ir}(\alpha)]$ ($\alpha \in [0, 1]$). When $h_i < 1$ an α-cut on $LMF(\tilde{G}^i)$ only exists for $\alpha \in [0, h_i]$ whereas an α-cut on $UMF(\tilde{G}^i)$ exists for all $\alpha \in [0, 1]$. One way to "extend" the α-cuts on $LMF(\tilde{G}^i)$ from $\alpha \in [0, h_i]$ to $\alpha \in [0, 1]$ is to define:

$$a'_{ir}(\alpha) = \begin{cases} a_{ir}(\alpha) & \alpha \le h_i \\ b_{ir}(\alpha) & \alpha > h_i \end{cases} \quad (6)$$

$$b'_{il}(\alpha) = \begin{cases} b_{il}(\alpha) & \alpha \le h_i \\ a_{il}(\alpha) & \alpha > h_i \end{cases} \quad (7)$$

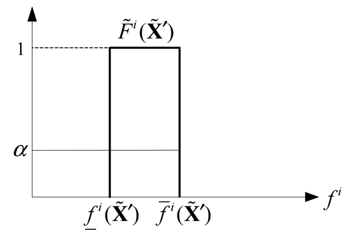

Fig. 3. $\tilde{F}^i(\tilde{\mathbf{X}}')$, the interpreted IT2 FS for firing interval $F^i(\tilde{\mathbf{X}}')$ of Rule-i.

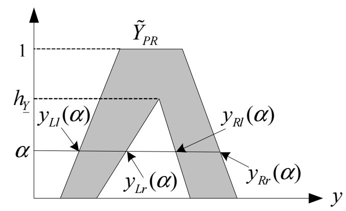

Fig. 4. $\tilde{Y}_{PR}$, the LWA for Perceptual Reasoning.

An interior FOU for $\tilde{Y}_{PR}$ is depicted in Fig. 4. The α-cut on $UMF(\tilde{Y}_{PR})$ is $[y_{Ll}(\alpha), y_{Rr}(\alpha)]$ and the α-cut on $LMF(\tilde{Y}_{PR})$ is $[y_{Lr}(\alpha), y_{Rl}(\alpha)]$, where the end-points of these α-cuts are computed as solutions to the following optimization problems [22], [23]:

$$y_{Ll}(\alpha) = \min_{\forall f^i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^m a_{il}(\alpha) f^i}{\sum_{i=1}^m f^i} \quad (8)$$

$$y_{Rr}(\alpha) = \max_{\forall f^i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^m b_{ir}(\alpha) f^i}{\sum_{i=1}^m f^i} \quad (9)$$

$$y_{Lr}(\alpha) = \min_{\forall f^i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^m a'_{ir}(\alpha) f^i}{\sum_{i=1}^m f^i} \quad (10)$$

$$y_{Rl}(\alpha) = \max_{\forall f^i \in [\underline{f}^i, \overline{f}^i]} \frac{\sum_{i=1}^m b'_{il}(\alpha) f^i}{\sum_{i=1}^m f^i} \quad (11)$$

Note that (see Fig. 4) $y_{Lr}(\alpha)$ and $y_{Rl}(\alpha)$ only exist for $\alpha \in [0, h_{\underline{Y}}]$; however, it is not possible to compute $h_{\underline{Y}}$ ahead of time; so, instead, the following simple strategy is used in [22], [23]: Compute $[y_{Lr}(\alpha), y_{Rl}(\alpha)]$ as though it exists, and IF $y_{Lr}(\alpha) \le y_{Rl}(\alpha)$, THEN keep $[y_{Lr}(\alpha), y_{Rl}(\alpha)]$; otherwise discard $[y_{Lr}(\alpha), y_{Rl}(\alpha)]$. The last value of α for which this test is passed is called $h_{\underline{Y}}$. Once $h_{\underline{Y}}$ is determined then it is no longer necessary to compute $[y_{Lr}(\alpha), y_{Rl}(\alpha)]$.

[7]We refer to (5) as "expressive" because it is not computed using multiplications, additions and divisions, as expressed by it. Instead, $LMF(\tilde{Y}_{PR})$ and $UMF(\tilde{Y}_{PR})$ are computed as explained in [22], [23].

[8]To connect our special LWA with the more general one in [22], [23], note that $c_{il}(\alpha) = c_{ir}(\alpha) = \underline{f}^i$ and $d_{il}(\alpha) = d_{ir}(\alpha) = \overline{f}^i$, where in [22], [23] the α-cut of $UMF(\tilde{W}^i) = [c_{il}(\alpha), d_{ir}(\alpha)]$ and the α-cut of $LMF(\tilde{W}^i) = [c_{ir}(\alpha), d_{il}(\alpha)]$, and $\tilde{W}^i$ plays the role of our F^i.

[9]In this notation, the first subscript is an index that runs from 1 to at most m, whereas the second subscript is a pneumonic for left or right.

With this as some background about the LWA as specialized to Perceptual Reasoning, we turn now to the verification of Assumption 1 (Section I) for $\tilde{Y}_{PR}$.

A. Perceptual Reasoning Verified

In this section we prove that $FOU(\tilde{Y}_{PR})$ *will resemble the three kinds of FOUs in a CWW codebook.* To begin, constructive tests are provided to establish when $FOU(\tilde{Y}_{PR})$ is an interior, left-shoulder or right-shoulder FOU.

Lemma 1: $FOU(\tilde{Y}_{PR})$ is an interior FOU if the $\alpha = 0$ α-cut of $\tilde{Y}_{PR}$ satisfies the following two inequalities (see Fig. 4):

$$y_{Ll}(0) < y_{Lr}(0) \quad (12)$$

$$y_{Rl}(0) < y_{Rr}(0) \quad \blacksquare \quad (13)$$

Lemma 2: $FOU(\tilde{Y}_{PR})$ is a left-shoulder FOU if the α-cuts of $\tilde{Y}_{PR}$ satisfy the following equality (compare Fig. 4 with Fig. 2(b)):

$$y_{Ll}(\alpha) = y_{Lr}(\alpha) = 0 \ \forall \alpha \in [0, 1] \quad \blacksquare \quad (14)$$

Lemma 3: $FOU(\tilde{Y}_{PR})$ is a right-shoulder FOU if the α-cuts of $\tilde{Y}_{PR}$ satisfy the following equality (compare Fig. 4 with Fig. 2(c)):

$$y_{Rl}(\alpha) = y_{Rr}(\alpha) = N \ \forall \alpha \in [0, 1] \quad \blacksquare \quad (15)$$

The tests in these lemmas are used in the proofs of Theorems 2-4, which are provided in [16].

Theorem 2. $FOU(\tilde{Y}_{PR})$ is an interior FOU if:

1) All $FOU(\tilde{G}^i)$ are interior FOUs, or
2) $FOU(\tilde{G}^i)$ consist of more than one kind of FOUs (e.g., interior and left-shoulder), and for at least two kinds there exists at least one associated firing interval for which $\underline{f}^i > 0$. $\blacksquare$

Condition (1) is intuitive because when it is true then $\tilde{Y}_{PR}$ is a weighted average of interior FOUs. Condition (2) is less intuitive; it allows for $\tilde{G}^i$ to be a mixture of just left and right shoulder FOUs, just left-shoulder and interior FOUs, just right-shoulder and interior FOUs, or interior, left-shoulder and right-shoulder FOUs. The proof of Theorem 1 considers all possible sub-cases. The condition that there must be at least one associated firing interval for which $\underline{f}^i > 0$ eliminates the possibility that $y_{Ll}(0) = y_{Lr}(0)$ and $y_{Rl}(0) = y_{Rr}(0)$.

Theorem 3. $FOU(\tilde{Y}_{PR})$ is a left-shoulder FOU if:

1) All $FOU(\tilde{G}^i)$ are left-shoulder FOUs, or
2) At least one $FOU(\tilde{G}^i)$ is a left-shoulder FOU, and for each $FOU(\tilde{G}^i)$ that is not a left-shoulder FOU its corresponding firing interval is such that $\underline{f}^i = 0$. $\blacksquare$

As in Theorem 2, Condition (1) of Theorem 2 is intuitive, because when it is true then $\tilde{Y}_{PR}$ is a weighted average of left-shoulder FOUs. Condition (2) is less intuitive; the condition that $\underline{f}^i = 0$ for all non-left-shoulder FOUs is needed so that (14) is satisfied.

Theorem 4. $FOU(\tilde{Y}_{PR})$ is a right-shoulder FOU if:

1) All $FOU(\tilde{G}^i)$ are right-shoulder FOUs, or
2) At least one $FOU(\tilde{G}^i)$ is a right-shoulder FOU, and for each $FOU(\tilde{G}^i)$ that is not a right-shoulder FOU its corresponding firing interval is such that $\underline{f}^i = 0$. $\blacksquare$

Comments about this theorem are so similar to those made for Theorem 3 that we leave them to the reader.

Theorem 5. When only one rule is fired, say the i^{th}, then $FOU(\tilde{Y}_{PR}) = FOU(\tilde{G}^i)$. $\blacksquare$

Such a result cannot be obtained for Mamdani Reasoning (see Fig. 5(a)) or other kinds of approximate reasoning methods. The fired-rule output set obtained from Mamdani reasoning leads to an FOU that is not in the CWW codebook.

Taken together, these four theorems demonstrate that $FOU(\tilde{Y}_{PR})$ *will resemble the three kinds of FOUs–left-, right-shoulder and interior FOUs–in a CWW codebook.* To the best knowledge of the authors, there is no other kind of reasoning (applied to IT2 FSs) available in the literature that can accomplish this.

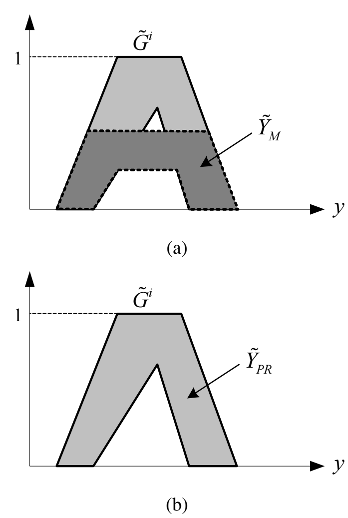

Fig. 5. When only one rule is fired: (a) Fired-rule FOU $\tilde{Y}_M$ from Mamdani Reasoning (minimum t-norm), and (b) $\tilde{Y}_{PR}$ from Perceptual Reasoning.

V. Conclusions

A new CWW engine, Perceptual Reasoning, has been proposed in this paper. It uses IF-THEN rules; however, different from traditional IF-THEN rules based CWW engines, which use Mamdani or TSK models, a LWA is used to combine the fired rules. The main advantage of perceptual reasoning is that its output FOU is a left-shoulder, right-shoulder, or an interior FOU, which resembles the three types of input FOUs in a CWW codebook. This is different from Mamdani and TSK models, none of which can map normal FSs into a normal FS. To the best knowledge of the authors, there is no other kind of reasoning (applied to IT2 FSs) available in the literature that can accomplish this.

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