Ordered Fuzzy Weighted Averages and Ordered Linguistic Weighted Averages

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***Abstract*— The ordered weighted average (OWA) operator has been widely used in decision-making. In many situations, however, providing crisp numbers for either the sub-criteria or the weights is problematic (there could be uncertainties about them), and it is more meaningful to provide intervals, type-1 fuzzy sets (T1 FSs), interval type-2 fuzzy sets (IT2 FSs), or a mixture of all of these, for the sub-criteria and weights. Two fuzzy extensions of the OWA, ordered fuzzy weighted averages for T1 FSs and ordered linguistic weighted averages for IT2 FSs, as wells as procedures for computing them, are introduced in this paper. They are compared with Zhou et al.'s T1 and IT2 fuzzy extensions of the OWA. Examples show that our extensions may give different results from Zhou et al.'s extensions when the legs of the FSs have intersections. Because our extensions coincide with the intuition of "FS in its entirety," they are the suggested ones to use.**

I. INTRODUCTION

The *weighted average* (WA) is arguably the earliest and still most widely used form of aggregation or fusion. We remind the reader of the well-known formula for the WA, i.e.,

$$y = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i}, \tag{1}$$

in which w_i are the weights (real numbers) that act upon the sub-criteria x_i (real numbers). In this paper, the term *sub-criteria* can mean data, features, decisions, recommendations, judgments, scores, etc. In (1), normalization is achieved by dividing the weighted numerator sum by the sum of all of the weights.

The *ordered weighted average* (OWA) operator [5], [16], [17], [21], [31], [32], [34]–[36], a generalization of the WA operator, was proposed by Yager to aggregate expert's opinions in decision making:

Definition 1: An OWA operator of dimension n is a mapping $y_{OWA} : R^n \to R$, which has an associated set of weights $\mathbf{w} = \{w_1, \ldots, w_n\}$ for which $w_i \in [0, 1]$, i.e.,

$$y_{OWA} = \frac{\sum_{i=1}^{n} w_i x_{\sigma(i)}}{\sum_{i=1}^{n} w_i}, \tag{2}$$

where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation function such that $\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\}$ are in descending order. □

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The key feature of the OWA operator is the ordering of the sub-criteria by value, a process that introduces a nonlinearity into the operation. It can be shown that the OWA operator is in the class of mean operators [4] as it is commutative, monotonic, and idempotent. It is also easy to see that for any $\mathbf{w}$, $\min_i x_i \le y_{OWA} \le \max_i x_i$.

The most attractive feature of the OWA operator is that it can implement different aggregation operators by setting the weights differently [5], e.g., by setting $w_i = 1/n$ it implements the mean operator, by setting $w_1 = 1$ and $w_i = 0$ $(i = 2, \ldots, n)$ it implements the maximum operator, by setting $w_i = 0$ $(i = 1, \ldots, n-1)$ and $w_n = 1$ it implements the minimum operator, and by setting $w_1 = w_n = 0$ and $w_i = 1/(n-2)$ it implements the so-called olympic aggregator, which is often used in obtaining aggregated scores from judges in olympic events such as gymnastics and diving.

Yager's original OWA operator [31] considers only crisp numbers. In many situations [20], [22], [28], however, providing crisp numbers for either the sub-criteria or the weights is problematic (there could be uncertainties about them), and it is more meaningful to provide intervals, type-1 fuzzy sets (T1 FSs), interval type-2 fuzzy sets (IT2 FSs) [18], or a mixture of all of these, for the sub-criteria and weights. Fuzzy extensions of the OWAs are the focus of this paper.

There has been many works on fuzzy extensions of OWAs, e.g., linguistic ordered weighted averaging [2], [8]–[10] and uncertain linguistic ordered weighted averaging [29]; however, for these extensions, only the sub-criteria are modeled as T1 FSs whereas the weights are still crisp numbers. To the authors' best knowledge, Zhou et al. [34]–[36] are the first to consider fuzzy weights.

This paper introduces new fuzzy extensions of the OWA called *ordered fuzzy weighted averages* (OFWAs) for T1 FSs and *ordered linguistic weighted averages* (OLWAs) for IT2 FSs, where both sub-criteria and weights are FSs, and compares them with Zhou et al.'s approaches.

The rest of this paper is organized as follows: Section II proposes the OFWA and OLWA. Section III introduces Zhou et al.'s T1 and IT2 fuzzy extensions of the OWA. Section IV compares the two categories of extensions. Finally, conclusions are drawn in Section V.

II. OUR FUZZY EXTENSIONS OF THE OWA

Our fuzzy extensions of the OWA, OFWAs and OLWAs, are introduced in this section. Because their computations use the *fuzzy weighted average* (FWA) [15], [20], [22], [28]

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and the *linguistic weighted average* (LWA) [20], [22], [23], [25], [28], the FWA and LWA are also introduced.

A. The Fuzzy Weighted Average (FWA)

Definition 2: [3], [6], [7], [13]–[15] An FWA is defined as

$$Y_{FWA} = \frac{\sum_{i=1}^{n} W_i X_i}{\sum_{i=1}^{n} W_i}, \tag{3}$$

where X_i and W_i are T1 FSs. □

(3) is an expressive way to describe the FWA, because Y_{FWA} is not computed using multiplications, additions or divisions, as expressed by (3). Instead, Y_{FWA}, which is also a T1 FS, is computed by using the α-cut decomposition theorem [12], [15]. Denote the α-cut on Y_{FWA} as $[y_L(\alpha), y_R(\alpha)]$, and the α-cut on X_i and W_i as $[a_i(\alpha), b_i(\alpha)]$ and $[c_i(\alpha), d_i(\alpha)]$, respectively, as shown in Fig. 1. Then [15],

$$y_L(\alpha) = \min_{\forall w_i(\alpha)\in[c_i(\alpha),d_i(\alpha)]} \frac{\sum_{i=1}^{n} a_i(\alpha) w_i(\alpha)}{\sum_{i=1}^{n} w_i(\alpha)} \tag{4}$$

$$y_R(\alpha) = \max_{\forall w_i(\alpha)\in[c_i(\alpha),d_i(\alpha)]} \frac{\sum_{i=1}^{n} b_i(\alpha) w_i(\alpha)}{\sum_{i=1}^{n} w_i(\alpha)} \tag{5}$$

$y_L(\alpha)$ and $y_R(\alpha)$ can be computed by the KM or EKM algorithms [11], [18], [22], [24], [27]. In summary, to compute Y_{FWA}:

1) For each $\alpha \in [0, 1]$, compute the corresponding α-cuts of the T1 FSs X_i and W_i, i.e., compute

$$X_i(\alpha) = [a_i(\alpha), b_i(\alpha)] \quad i = 1, ..., n \tag{6}$$

$$W_i(\alpha) = [c_i(\alpha), d_i(\alpha)] \quad i = 1, ..., n \tag{7}$$

2) For each $\alpha \in [0, 1]$, compute $y_L(\alpha)$ in (4) and $y_R(\alpha)$ in (5) using the KM or EKM Algorithms.
3) Connect all left-coordinates $(y_L(\alpha), \alpha)$ and all right-coordinates $(y_R(\alpha), \alpha)$ to form the T1 FS Y_{FWA}.

B. The Ordered Fuzzy Weighted Average (OFWA)

As its name suggests, the OFWA is a combination of the OWA and the FWA.

Definition 3: An OFWA is defined as

$$Y_{OFWA} = \frac{\sum_{i=1}^{n} W_i X_{\sigma(i)}}{\sum_{i=1}^{n} W_i}, \tag{8}$$

where X_i and W_i are T1 FSs, and $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation function such that $\{X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}\}$ are in descending order. □

Definition 4: A group of T1 FSs $\{X_i\}_{i=1}^{n}$ are in descending order if $X_i \succeq X_j$ for $\forall i < j$ by a ranking method. □

Any T1 FS ranking method can be used to find σ. In this paper, Yager's first method [30] is used, which first computes the centroid of each T1 FS and then ranks them to obtain the order of the corresponding T1 FSs.

To compute Y_{OFWA}, X_i are first sorted in descending order and called by the same name, but now $X_1 \succeq X_2 \succeq \cdots \succeq X_n$ (W_i are not changed during this step); then, the FWA algorithm introduced in Section II-A can be used to compute Y_{OFWA}.

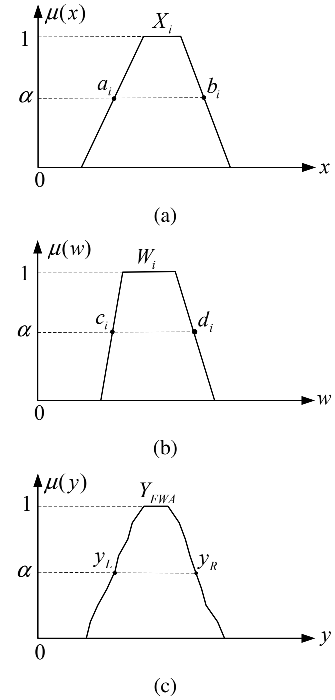

Fig. 1. α-cuts on (a) X_i, (b) W_i, and, (c) Y_{FWA}.

C. The Linguistic Weighted Average (LWA)

Definition 5: [20], [22], [23], [25], [28] An LWA is defined as

$$\tilde{Y}_{LWA} = \frac{\sum_{i=1}^{n} \tilde{W}_i \tilde{X}_i}{\sum_{i=1}^{n} \tilde{W}_i}, \tag{9}$$

where $\tilde{X}_i$ and $\tilde{W}_i$ are IT2 FSs. □

Again, (9) is an expressive way to summarize the LWA. It has been shown [20], [22], [23], [25] that computing an LWA is equivalent to computing two FWAs. Denote the lower membership function (LMF) of $\tilde{Y}_{LWA}$ as $\underline{Y}_{LWA}$, and the upper membership function (UMF) of $\tilde{Y}_{LWA}$ as $\overline{Y}_{LWA}$. Using the notations shown in Fig. 2, the α-cuts on $\overline{Y}_{LWA}$ and $\underline{Y}_{LWA}$ are computed as:

$$y_{Ll}(\alpha) = \min_{w_i(\alpha)\in[c_{il}(\alpha),d_{ir}(\alpha)]} \frac{\sum_{i=1}^{n} a_{il}(\alpha) w_i(\alpha)}{\sum_{i=1}^{n} w_i(\alpha)}, \alpha \in [0, 1] \tag{10}$$

$$y_{Rr}(\alpha) = \max_{w_i(\alpha)\in[c_{il}(\alpha),d_{ir}(\alpha)]} \frac{\sum_{i=1}^{n} b_{ir}(\alpha) w_i(\alpha)}{\sum_{i=1}^{n} w_i(\alpha)}, \alpha \in [0, 1] \tag{11}$$

$$y_{Lr}(\alpha) = \min_{w_i(\alpha)\in[c_{ir}(\alpha),d_{il}(\alpha)]} \frac{\sum_{i=1}^{n} a_{ir}(\alpha) w_i(\alpha)}{\sum_{i=1}^{n} w_i(\alpha)}, \alpha \in [0, h_{\min}] \tag{12}$$

$$y_{Rl}(\alpha) = \max_{w_i(\alpha)\in[c_{ir}(\alpha),d_{il}(\alpha)]} \frac{\sum_{i=1}^{n} b_{il}(\alpha) w_i(\alpha)}{\sum_{i=1}^{n} w_i(\alpha)}, \alpha \in [0, h_{\min}] \tag{13}$$

where

$$h_{\min} = \min\{\min_{\forall i} h_{\underline{X}_i}, \min_{\forall i} h_{\underline{W}_i}\} \tag{14}$$

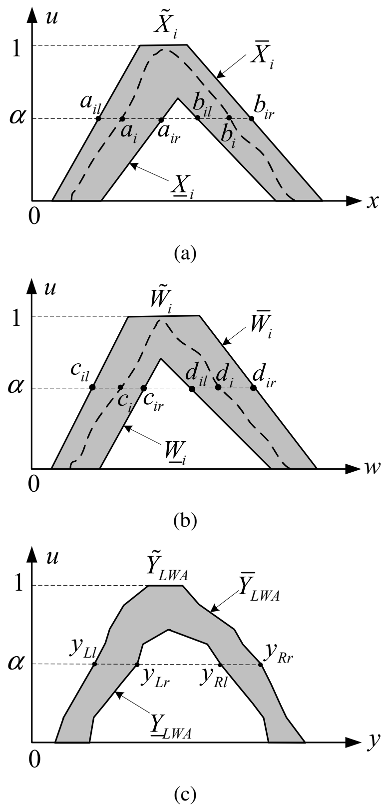

Fig. 2. α-cuts on (a) $\tilde{X}_i$, (b) $\tilde{W}_i$, and, (c) $\tilde{Y}_{LWA}$.

in which $h_{\underline{X}_i}$ is the height of the LMF of $\tilde{X}_i$ [see Fig. 2(a)], and $h_{\underline{W}_i}$ is the height of the LMF of $\tilde{W}_i$ [see Fig. 2(b)].

The procedure to compute $\overline{Y}_{LWA}$ is:

1) Select appropriate m α-cuts for $\overline{Y}_{LWA}$ (e.g., divide [0, 1] into $m-1$ intervals and set $\alpha_j = (j-1)/(m-1)$, $j = 1, 2, ..., m$).
2) For each α_j, find the corresponding α-cuts $[a_{il}(\alpha_j), b_{ir}(\alpha_j)]$ and $[c_{il}(\alpha_j), d_{ir}(\alpha_j)]$ on $\overline{X}_i$ and $\overline{W}_i$ ($i = 1, ..., n$). Use a KM or EKM algorithm to find $y_{Ll}(\alpha_j)$ in (10) and $y_{Rr}(\alpha_j)$ in (11).
3) Connect all left-coordinates $(y_{Ll}(\alpha_j), \alpha_j)$ and all right-coordinates $(y_{Rr}(\alpha_j), \alpha_j)$ to form the T1 FS $\overline{Y}_{LWA}$.

To compute $\underline{Y}_{LWA}$:

1) Determine $h_{\underline{X}_i}$ and $h_{\underline{W}_i}$, $i = 1, \ldots, n$, and $h_{\min}$ in (14).
2) Select appropriate p α-cuts for $\underline{Y}_{LWA}$ (e.g., divide $[0, h_{\min}]$ into $p-1$ intervals and set $\alpha_j = h_{\min}(j-1)/(p-1)$, $j = 1, 2, ..., p$).
3) For each α_j, find the corresponding α-cuts $[a_{ir}(\alpha_j), b_{il}(\alpha_j)]$ and $[c_{ir}(\alpha_j), d_{il}(\alpha_j)]$ on $\underline{X}_i$ and $\underline{W}_i$. Use a KM or EKM algorithm to find $y_{Lr}(\alpha_j)$ in (12) and $y_{Rl}(\alpha_j)$ in (13).
4) Connect all left-coordinates $(y_{Lr}(\alpha_j), \alpha_j)$ and all right-coordinates $(y_{Rl}(\alpha_j), \alpha_j)$ to form the T1 FS $\underline{Y}_{LWA}$.

D. The Ordered Linguistic Weighted Average (OLWA)

As its name suggests, the OLWA is a combination of the OWA and the LWA.

Definition 6: An OLWA is defined as

$$\tilde{Y}_{OLWA} = \frac{\sum_{i=1}^{n} \tilde{W}_i \tilde{X}_{\sigma(i)}}{\sum_{i=1}^{n} \tilde{W}_i}, \tag{15}$$

where $\tilde{X}_i$ and $\tilde{W}_i$ are IT2 FSs, $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation function such that $\{\tilde{X}_{\sigma(1)}, \ldots, \tilde{X}_{\sigma(n)}\}$ are in descending order. □

Definition 7: A group of IT2 FSs $\{\tilde{X}_i\}_{i=1}^n$ are in descending order if $\tilde{X}_i \succeq \tilde{X}_j$ for $\forall i < j$ by a ranking method. □

Any IT2 FS ranking method can be used to find σ. In this paper, the centroid-based ranking method [20], [22], [26] is used, which first computes the center of centroid of each IT2 FS and then ranks them to obtain the order of the corresponding IT2 FSs.

To compute the OLWA, all $\tilde{X}_i$ are first sorted in descending order and called by the same name, but now $\tilde{X}_1 \succeq \tilde{X}_2 \succeq \cdots \succeq \tilde{X}_n$ (note that $\tilde{W}_i$ are not changed during this step); then, the LWA algorithm introduced in Section II-C can be used to compute the OLWA.

III. Zhou et al.'s Fuzzy Extensions of the OWA

Zhou, et al. [34]–[36] were the first to consider fuzzy weights in the OWAs. Their approaches are introduced in this section.

A. T1 Fuzzy OWAs (T1FOWAs)

Zhou et al. [35], [36] defined a T1 fuzzy OWA (T1FOWA) as:

Definition 8: Given T1 FSs $\{W_i\}_{i=1}^n$ and $\{X_i\}_{i=1}^n$, the membership function of a T1FOWA is computed by (16) on top of the next page, where $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation function such that $\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\}$ are in descending order. □

$\mu_{Y_{T1FOWA}}(y)$ can be understood from the Extension Principle [33], i.e., first all combinations of w_i and x_i whose OWA is y are found, and for the j^{th} combination, the resulting y_j has a membership grade $\mu(y_j)$ which is the minimum of the corresponding $\mu_{X_i}(x_i)$ and $\mu_{W_i}(w_i)$. Then, $\mu_{Y_{T1FOWA}}(y)$ is the maximum of all these $\mu(y_j)$.

Y_{T1FOWA} can be computed efficiently using α-cuts [34], similar to the way they are used in computing the FWA. Denote $Y_{T1FOWA}(\alpha) = [y'_L(\alpha), y'_R(\alpha)]$ and use the same notations for α-cuts on X_i and W_i as in Fig. 1. Then,

$$y'_L(\alpha) = \min_{\forall w_i(\alpha) \in [c_i(\alpha), d_i(\alpha)]} \frac{\sum_{i=1}^{n} a_{\sigma(i)}(\alpha) w_i(\alpha)}{\sum_{i=1}^{n} w_i(\alpha)} \tag{17}$$

$$y'_R(\alpha) = \max_{\forall w_i(\alpha) \in [c_i(\alpha), d_i(\alpha)]} \frac{\sum_{i=1}^{n} b_{\sigma(i)}(\alpha) w_i(\alpha)}{\sum_{i=1}^{n} w_i(\alpha)} \tag{18}$$

$y'_L(\alpha)$ and $y'_R(\alpha)$ can also be computed using KM or EKM algorithms.

Observe from (17) and (18) that Zhou et al. consider each α-cut on X_i individually, and the permutation function $\sigma(i)$ may change from one α-cut to another α-cut. Prior to Zhou

$$\mu_{Y_{T1FOWA}}(y) = \sup_{\frac{\sum_{i=1}^n w_i x_{\sigma(i)}}{\sum_{i=1}^n w_i} = y} \min(\mu_{W_1}(w_1), \cdots, \mu_{W_n}(w_n), \mu_{X_1}(x_1), \cdots, \mu_{X_n}(x_n)) \tag{16}$$

et al.'s approach, no one had ever defined a ranking method for FSs (either T1 or IT2) that may change the ranking of the FSs from one α-cut to the next. Furthermore, even for the same α-cut, (17) and (18) may use different $\sigma(i)$. Consider a simple example $[a_1(\alpha), b_1(\alpha)] = [2, 3]$ and $[a_2(\alpha), b_2(\alpha)] = [1, 4]$. Then, in computing $y'_L(\alpha)$, $a_1(\alpha)$ is associated with $w_1(\alpha)$ and $a_2(\alpha)$ is associated with $w_2(\alpha)$; however, in computing $y'_R(\alpha)$, $b_2(\alpha)$ is associated with $w_1(\alpha)$ and $b_1(\alpha)$ is associated with $w_2(\alpha)$, i.e., the points from the same α-cut on X_i have different rankings and hence different weights in computing $y'_L(\alpha)$ and $y'_R(\alpha)$, which sounds counter-intuitive.

Generally the OFWA and the T1FOWA give different outputs, as indicated by the following:

Theorem 1: The OFWA and the T1FOWA have different results when at least one of the following two conditions occurs:

1) The left leg of X_i intersects the left leg of X_j, $i \neq j$.
2) The right leg of X_i intersects the right leg of X_j, $i \neq j$. □

Proof: Because the proof for Condition 2 is very similar to that for Condition 1, only the proof for Condition 1 is given here.

Assume the left leg of X_i intersects the left leg of X_j at $\alpha = \lambda \in (0, 1)$, as shown in Fig. 3. Then, $a_i(\alpha) > a_j(\alpha)$ when $\alpha \in [0, \lambda)$ and $a_i(\alpha) < a_j(\alpha)$ when $\alpha \in (\lambda, 1]$.

For an $\alpha_1 \in [0, \lambda)$, $y'_L(\alpha_1)$ in (17) is computed as

$$y'_L(\alpha_1) = \min_{\forall w_i(\alpha_1) \in [c_i(\alpha_1), d_i(\alpha_1)]} \frac{\sum_{i=1}^n a_{\sigma_1(i)}(\alpha_1) w_i(\alpha_1)}{\sum_{i=1}^n w_i(\alpha_1)} \tag{19}$$

where $\sigma_1 : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation function such that $\{a_{\sigma_1(1)}(\alpha_1), x_{\sigma_1(2)}(\alpha_1), \ldots, x_{\sigma_1(n)}(\alpha_1)\}$ are in descending order. Because $a_i(\alpha_1) > a_j(\alpha_1)$, it follows that $\sigma_1(i) < \sigma_1(j)$.

For an $\alpha_2 \in (\lambda, 1]$, $y'_L(\alpha_2)$ in (17) is computed as

$$y'_L(\alpha_2) = \min_{\forall w_i(\alpha_2) \in [c_i(\alpha_2), d_i(\alpha_2)]} \frac{\sum_{i=1}^n a_{\sigma_2(i)}(\alpha_2) w_i(\alpha_2)}{\sum_{i=1}^n w_i(\alpha_2)} \tag{20}$$

where $\sigma_2 : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation function such that $\{a_{\sigma_2(1)}(\alpha_2), a_{\sigma_2(2)}(\alpha_2), \ldots, a_{\sigma_2(n)}(\alpha_2)\}$ are in descending order. Because $a_i(\alpha_2) < a_j(\alpha_2)$, it follows that $\sigma_2(i) > \sigma_2(j)$, i.e., $\sigma_1 \neq \sigma_2$.

On the other hand, for Y_{OFWA}, no matter which ranking method is used, the permutation function σ is the same for all $\alpha \in [0, 1]$. Without loss of generality, assume $X_j \succeq X_i$ by a ranking method. Then, in (15) $\sigma(i) > \sigma(j)$, and, for any $\alpha \in [0, 1]$, $y_L(\alpha)$ is computed by (5).

Clearly, for any $\alpha \in [0, \lambda)$, $y_L(\alpha) \neq y'_L(\alpha)$ because $\sigma \neq \sigma_1$. Consequently, the left legs of Y_{OFWA} and Y_{T1FOWA} are different. □

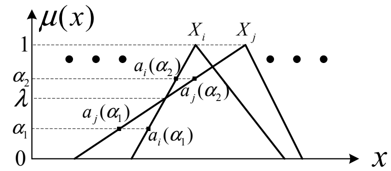

Fig. 3. Illustration of intersecting X_i and X_j, where the left leg of X_i intersects the left leg of X_j.

Generally, the intersections occur when an X_i is much wider (more uncertain) that its neighbors. Observe two important points from Theorem 1:

1) Only the intersection of a left leg with another left leg, or a right leg with another right leg, would definitely lead to different Y_{T1FOWA} and Y_{OFWA}. The intersection of a left leg with a right leg does not lead to different Y_{T1FOWA} and Y_{OFWA}.
2) Only the intersections of X_i may lead to different Y_{T1FOWA} and Y_{OFWA}. The intersections of W_i have no effect on this because the permutation function σ does not depend on W_i.

B. IT2 Fuzzy OWAs (IT2FOWAs)

Zhou et al. [36] defined the IT2 fuzzy OWA (IT2FOWA) as:

Definition 9: Given IT2 FSs $\{\tilde{W}_i\}_{i=1}^n$ and $\{\tilde{X}_i\}_{i=1}^n$, the membership function of an IT2FOWA is computed by (21) on top of the next page, where W_i^e and X_i^e are embedded T1 FSs of $\tilde{W}_i$ and $\tilde{X}_i$, respectively, and $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation function such that $\{x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\}$ are in descending order. □

Comparing (21) with (16), observe that the bracketed term in (21) is a T1FOWA, and the IT2FOWA is the union of all possible T1FOWAs computed from the embedded T1 FSs of $\tilde{X}_i$ and $\tilde{W}_i$. The Wavy Slice Representation Theorem [19] for IT2 FSs is used implicitly in this definition.

$\tilde{Y}_{IT2FOWA}$ can be computed efficiently using α-cuts, similar to the way they were used in computing the LWA. Denote the α-cut on the UMF of $\tilde{Y}_{IT2FOWA}$ as $\overline{Y}_{IT2FOWA}(\alpha) = [y'_{Ll}(\alpha), y'_{Rr}(\alpha)]$ for $\forall \alpha \in [0, 1]$, the α-cut on the LMF of $\tilde{Y}_{IT2FOWA}$ as $\underline{Y}_{IT2FOWA}(\alpha) = [y'_{Lr}(\alpha), y'_{Rl}(\alpha)]$ for $\forall \alpha \in [0, h_{\min}]$, where $h_{\min}$ is defined in (14). Using the same notations for α-cuts on $\tilde{X}_i$ and $\tilde{W}_i$ as in Fig. 2, it is

$$\mu_{\tilde{Y}_{IT2FOWA}}(y) = \bigcup_{\forall W_i^e, X_i^e} \left[\sup_{\frac{\sum_{i=1}^n w_i x_{\sigma(i)}}{\sum_{i=1}^n w_i} = y} \min(\mu_{W_1^e}(w_1), \cdots, \mu_{W_n^e}(w_n), \mu_{X_1^e}(x_1), \cdots, \mu_{X_n^e}(x_n)) \right] \tag{21}$$

easy to show that

$$y'_{Ll}(\alpha) = \min_{\forall w_i(\alpha) \in [c_{il}(\alpha), d_{ir}(\alpha)]} \frac{\sum_{i=1}^n a_{\sigma(i),l}(\alpha) w_i(\alpha)}{\sum_{i=1}^n w_i(\alpha)}, \quad \alpha \in [0,1] \tag{22}$$

$$y'_{Rr}(\alpha) = \max_{\forall w_i(\alpha) \in [c_{il}(\alpha), d_{ir}(\alpha)]} \frac{\sum_{i=1}^n b_{\sigma(i),r}(\alpha) w_i(\alpha)}{\sum_{i=1}^n w_i(\alpha)}, \quad \alpha \in [0,1] \tag{23}$$

$$y'_{Lr}(\alpha) = \min_{\forall w_i(\alpha) \in [c_{ir}(\alpha), d_{il}(\alpha)]} \frac{\sum_{i=1}^n a_{\sigma(i),r}(\alpha) w_i(\alpha)}{\sum_{i=1}^n w_i(\alpha)}, \quad \alpha \in [0, h_{\min}] \tag{24}$$

$$y'_{Rl}(\alpha) = \max_{\forall w_i(\alpha) \in [c_{ir}(\alpha), d_{il}(\alpha)]} \frac{\sum_{i=1}^n b_{\sigma(i),l}(\alpha) w_i(\alpha)}{\sum_{i=1}^n w_i(\alpha)}, \quad \alpha \in [0, h_{\min}] \tag{25}$$

$y'_{Ll}(\alpha)$, $y'_{Rr}(\alpha)$, $y'_{Lr}(\alpha)$ and $y'_{Rl}(\alpha)$ can also be computed using KM or EKM algorithms.

As the IT2FOWA also considers each α-cut separately in ranking, it has the same property as the T1FOWA, i.e., the ranking of a FS may change within an α-cut and between different α-cuts. Similarly, generally the OLWA and the IT2FOWA give different outputs, as indicated by the following:

Theorem 2: The OLWA and the IT2FOWA have different results when at least one of the following four conditions occurs:

1) The left leg of $\overline{X}_i$ intersects the left leg of $\overline{X}_j$, $i \neq j$.
2) The left leg of $\underline{X}_i$ intersects the left leg of $\underline{X}_j$, $i \neq j$.
3) The right leg of $\overline{X}_i$ intersects the right leg of $\overline{X}_j$, $i \neq j$.
4) The right leg of $\underline{X}_i$ intersects the right leg of $\underline{X}_j$, $i \neq j$. □

The correctness of Theorem 2 can be easily seen from Theorem 1, i.e., Condition 1 leads to different $y_{Ll}(\alpha)$ and $y'_{Ll}(\alpha)$ for certain α, Condition 2 leads to different $y_{Lr}(\alpha)$ and $y'_{Lr}(\alpha)$ for certain α, Condition 3 leads to different $y_{Rr}(\alpha)$ and $y'_{Rr}(\alpha)$ for certain α, and Condition 4 leads to different $y_{Rl}(\alpha)$ and $y'_{Rl}(\alpha)$ for certain α.

Generally, the intersections occur when an $\tilde{X}_i$ is much wider (more uncertain) that its neighbors. Observe two important points from Theorem 2:

1) Only the intersection of a left leg of $\tilde{X}_i$ with a left leg of $\tilde{X}_j$ ($i \neq j$), or a right leg of $\tilde{X}_i$ with a right leg of $\tilde{X}_j$, will definitely lead to different $\tilde{Y}_{IT2FOWA}$ and $\tilde{Y}_{OLWA}$. The intersection of a left leg of $\tilde{X}_i$ with a right leg of $\tilde{X}_j$ will not lead to different $\tilde{Y}_{IT2FOWA}$ and $\tilde{Y}_{OLWA}$.
2) The intersections of $\tilde{W}_i$ have no effects on whether or not $\tilde{Y}_{IT2FOWA}$ and $\tilde{Y}_{OLWA}$ are different because the permutation function σ does not depend on $\tilde{W}_i$.

IV. Comparative Studies and Discussions

Comparatives studies of our fuzzy extensions of the OWAs with Zhou et al.'s approaches are presented in this section, followed by discussions about the two kinds of extensions.

A. Comparative Studies

The following example illustrates the difference between FWA, OFWA and T1FOWA.

Example 1: Consider X_i and W_i shown in Figs. 4(a) and 4(b). The corresponding Y_{FWA} is shown in Fig. 4(c) as the solid curve, Y_{OFWA} the dashed curve, and Y_{T1OFWA} the dotted curve. Note that all three results are different. The difference between Y_{OFWA} and Y_{T1FOWA} is caused by the facts that the left leg of X_3 crosses the left legs of X_1 and X_4, and the right leg of X_3 crosses the right leg of X_2, which cause the permutation function σ to change as α increases [e.g., to begin, the left leg of X_3 lies to the left of the left leg of X_4, but at a certain value of α (where the two left legs intersect), there is a reversal of this relationship]. There will be no differences between Y_{OFWA} and Y_{T1FOWA} if X_i do not have such kinds of intersections. □

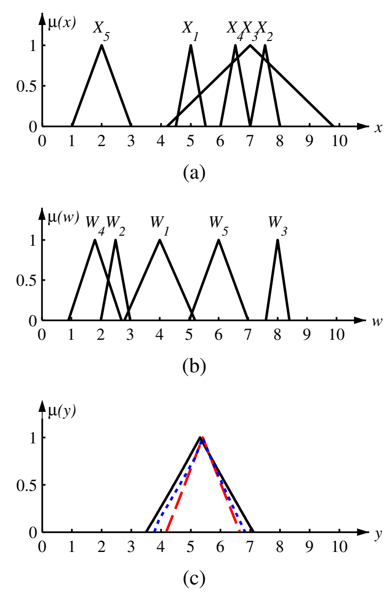

Fig. 4. Example 1: (a) X_i, (b) W_i, and, (c) Y_{FWA} (solid curve), Y_{OFWA} (dashed curve) and Y_{T1FOWA} (dotted curve).

The following example illustrates the difference between LWA, OLWA and IT2FOWA.

Example 2: Consider $\tilde{X}_i$ and $\tilde{W}_i$ shown in Figs. 5(a) and 5(b), respectively. The corresponding $\tilde{Y}_{LWA}$ is shown in Fig. 5(c) as the solid curve, $\tilde{Y}_{OLWA}$ the dashed curve, and $\tilde{Y}_{IT2OFWA}$ the dotted curve. Note that all three results are different. The difference between $\tilde{Y}_{OLWA}$ and $\tilde{Y}_{IT2FOWA}$ is caused by the facts that the left leg of $\underline{X}_3$ ($\overline{X}_3$) crosses the left legs of $\underline{X}_1$ and $\underline{X}_4$ ($\overline{X}_1$ and $\overline{X}_4$), and the right leg of $\underline{X}_3$ ($\overline{X}_3$) crosses the right leg of $\underline{X}_2$ ($\overline{X}_2$), which cause the permutation function σ to change as α increases. There will be no differences between $\tilde{Y}_{OLWA}$ and $\tilde{Y}_{IT2FOWA}$ if $\tilde{X}_i$ do not have such kinds of intersections. □

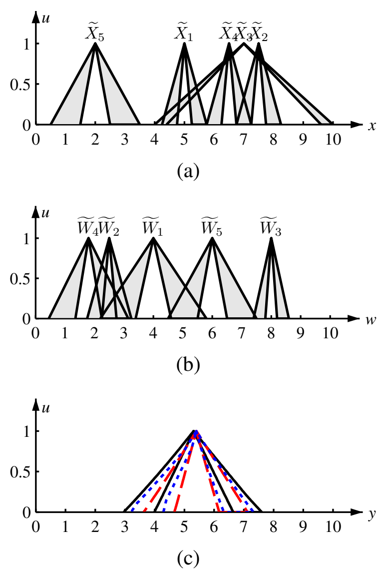

Fig. 5. Example 2: (a) $\tilde{X}_i$, (b) $\tilde{W}_i$, and, (c) $\tilde{Y}_{LWA}$ (solid curve), $\tilde{Y}_{OLWA}$ (dashed curve) and $\tilde{Y}_{IT2FOWA}$ (dotted curve).

B. Discussions

The T1 and IT2 fuzzy OWAs have been derived by considering each α-cut separately, whereas the OFWA and OLWA have been derived by considering each sub-criterion as a whole. Generally the two approaches give different results. Then, a natural question is: Which approach should be used in practice?

We believe that it is more intuitive to consider an FS in its entirety during ranking of FSs. To the best of our knowledge [26], all ranking methods based on α-cuts deduce a single number to represent each FS and then sort these numbers to obtain the ranks of the FSs. Each of these numbers is computed based only on the FS under consideration, i.e., no α-cuts on other FSs to be ranked are considered. Because in OFWA and OLWA the FSs are first ranked and then the WAs are computed, they coincide with our "FS in its entirety" intuition, and hence they are recommended in this paper.

Interestingly, this "FS in its entirety" intuition was also used implicitly in developing linguistic ordered weighted averaging [8], uncertain linguistic ordered weighted averaging [29], and fuzzy linguistic ordered weighted averaging [1].

Finally, note that the OFWA can be viewed as a special case of the FWA, and the OLWA can be viewed as a special case of the LWA.

V. Conclusions

In this paper, two fuzzy extensions of the OWA, as well as procedures for computing them, have been introduced, namely, ordered fuzzy weighted averages for T1 FSs and ordered linguistic weighted averages for IT2 FSs. They were compared with Zhou et al.'s T1 and IT2 fuzzy extensions of the OWA. Examples showed that our extensions may give different results from Zhou et al.'s extensions when the legs of the FSs have intersections. Because our extensions coincide with the "FS in its entirety" intuition (i.e., they do not require the use of different rankings for FSs for different values of α), they are the suggested ones to use.

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