On the Continuity of Type-1 and Interval Type-2 Fuzzy Logic Systems

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***Abstract*—This paper studies the continuity of the input–output mappings of fuzzy logic systems (FLSs), including both type-1 (T1) and interval type-2 (IT2) FLSs. We show that a T1 FLS being an universal approximator is equivalent to saying that a T1 FLS has a continuous input–output mapping. We also derive the condition under which a T1 FLS is discontinuous. For IT2 FLSs, we consider six type-reduction and defuzzification methods (the Karnik–Mendel method, the uncertainty bound method, the Wu–Tan method, the Nie–Tan method, the Du–Ying method, and the Begian–Melek–Mendel method) and derive the conditions under which continuous and discontinuous input–output mappings can be obtained. Guidelines for designing continuous IT2 FLSs are also given. This paper is to date the most comprehensive study on the continuity of FLSs. Our results will be very useful in the selection of the parameters of the membership functions to achieve a desired continuity (e.g., for most traditional modeling and control applications) or discontinuity (e.g., for hybrid and switched systems modeling and control).**

***Index Terms*—Continuity, discontinuity, fuzzy logic modeling and control, hybrid and switched systems, interval type-2 fuzzy logic systems (FLSs), monotonicity, smoothness.**

I. Introduction

MODELING and control is the most widely used application of both type-1 (T1) fuzzy logic systems (FLSs) [3], [10], [22], [52], [65], [66] and interval type-2 (IT2) FLSs [20], [21], [34], [42], [50], [61]–[63]. Essentially, an FLS implements a function representing a mapping between inputs and outputs. For modeling, an FLS represents the relationship between the inputs and outputs of a system, e.g., the force applied to the acceleration pedal of a car and its acceleration. For control, usually the inputs are signals related to the errors (e.g., the *error* between the desired speed and the car's actual speed, and/or the *change of error*, and/or the *integral of the error*), and the outputs are the control signals that will be applied to the plant under control (e.g., the force that should be applied to the acceleration pedal) to reduce such errors; therefore, the FLS implements a control law.

In many cases, continuous and smooth input–output mapping is desired for an FLS, because most physical systems are continuous, and a continuous and smooth control surface is usually more favorable in terms of stability and performance, e.g., Wu and Tan [59], [61], [62] and Jammeh *et al.* [21] have shown that an IT2 fuzzy logic controller may outperform its T1 counterpart because it gives a smoother control surface, especially in the region around the steady state (both the *error* and the *change of error* approach 0). Therefore, for such applications, we need to avoid abrupt changes, especially discontinuities, in the input–output mappings. It would be very beneficial to find the conditions under which an FLS gives a continuous input–output mapping so that we can ensure a continuous mapping when it is desired.

In this paper, we distinguish between continuity and smoothness. A continuous function is defined in Section II-A. A smooth function requires the continuity of not only itself, but also its derivatives. Therefore, a smooth function is more difficult to obtain than a continuous function. The continuity of FLSs is studied in this paper. The smoothness of FLSs will be investigated in the future.

Surprisingly, though fuzzy sets have been used for more than 40 years [67], little research has been conducted directly on the continuity of FLSs. Many results have shown that T1 FLSs are universal approximators [8], [9], [11], [27], [29], [30], [51]. This is equivalent to saying that a T1 FLS can implement any real continuous function, as we will prove in this paper; however, it is still unclear whether and when a T1 FLS can implement a discontinuous real function. Furthermore, to the best of the authors' knowledge, no researcher has considered the continuity of IT2 FLSs.

The rest of this paper is organized as follows: Section II studies the continuity of T1 FLSs. Section III studies the continuity of IT2 FLSs with Karnik–Mendel type-reduction and center-of-sets defuzzification, the most popular IT2 FLSs in practice. Section IV studies the continuity of IT2 FLSs with uncertainty-bound type-reduction and center-of-sets defuzzification. Section V studies the continuity of IT2 FLSs using another four methods, which combine type-reduction and defuzzification. Section VI summarizes the continuity of IT2 FLSs with different configurations and proposes design guidelines for continuous IT2 FLSs and several new research directions. Finally, Section VII draws conclusions. The proofs for all theorems are given in the Appendix.

II. Continuity of T1 Fuzzy Logic Systems

This section studies the continuity of T1 FLSs. First, properties of continuous functions are reviewed.

Manuscript received February 3, 2010; revised July 8, 2010; accepted November 1, 2010. Date of publication November 11, 2010; date of current version February 7, 2011.

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Digital Object Identifier 10.1109/TFUZZ.2010.2091962

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A. Properties of Continuous Functions

Definition 1: A single-variable function $f(x)$ is *continuous* at c if and only if $f(x)$ is defined at c, and $\lim_{x\to c} f(x) = f(c)$, i.e., for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$.

Recall the following facts about continuous functions from elementary calculus [12], [24].

1) If $f(x)$ is differentiable at c, then it is continuous at c.
2) Suppose both $f_1(x)$ and $f_2(x)$ are continuous at c:
   a) For any constant k, the function $k \cdot f_1(x)$ is continuous at c.
   b) $f_1(x) + f_2(x)$ is continuous at c.
   c) $f_1(f_2(x))$ is continuous at c if $f_1(x)$ is continuous at $f_2(c)$.
   d) $f_1(x)/f_2(x)$ is continuous at c if $f_2(c) \neq 0$.

We distinguish between two types of discontinuities in this paper.

Definition 2: A function $f(x)$ has a *gap discontinuity* at c if $f(c)$ is *undefined*.

For example, $f_1(x)/f_2(x)$ has a gap discontinuity at c if $f_2(c) = 0$.

Definition 3: A function $f(x)$ has a *jump discontinuity* at c if $f(c)$ is *defined* but $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$, i.e., both $f(c)$ and $f(c+\delta)$ are defined, but $f(c+\delta)$ does not approach $f(c)$ as δ approaches 0.

For example, $f(x) = \begin{cases} 2, & x<0 \\ 3, & x\geq 0 \end{cases}$ has a jump discontinuity at $x = 0$.

A multivariable continuous function $f(\mathbf{x})$ of an M-dimension input $\mathbf{x} = (x_1, x_2, \ldots, x_M)$ is defined as follows.

Definition 4: A multivariable function $f(\mathbf{x})$ is *continuous* at $\mathbf{c} = (c_1, c_2, \ldots, c_M)$ if and only if it is defined at $\mathbf{c}$, and for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\max_{m=1,\ldots,M} |x_m - c_m| < \delta \Rightarrow |f(\mathbf{x}) - f(\mathbf{c})| < \epsilon$.

The facts about continuous single-variable functions, which were introduced after Definition 1, also hold for continuous multivariable functions.

B. Structure of the T1 FLSs

For simplicity, we consider only multiantecedent single-consequent T1 FLSs in this paper; however, our results can be easily extended to multiantecedent multiconsequent T1 FLSs, because the latter can be decomposed into several multi-antecedent single-consequent T1 FLSs [31].

The T1 FLS has M inputs, $\{x_m\}_{m=1,2,\ldots,M}$, and one output, y. Assume the mth input has N_m membership functions (MFs) in its universe of discourse, $\mathbb{X}_m$. Denote the nth MF in the m^{th} input domain as X_{mn}. A complete rulebase with all possible combinations of the input MFs consists of $K = \prod_{m=1}^{M} N_m$ rules in the form of

$$R^k: \text{ IF } x_1 \text{ is } X_{1,n_{1k}} \text{ and} \ldots \text{ and } x_M \text{ is } X_{M,n_{Mk}}$$
$$\text{THEN } y \text{ is } y_k, \quad n_{ik} = 1, 2, \ldots, N_i, k = 1, 2, \ldots, K$$

where y_k is a constant, and generally it is different for different rules. An example rulebase for a T1 FLS with two inputs ($M = 2$) and three MFs for each input ($N_1 = N_2 = 3$) is shown in Table I. Note that this T1 FLS can be viewed as the simplest TSK model, where each rule consequent is represented by a crisp number. It can also be viewed as a Mamdani model with height defuzzification [42], i.e., y_k represents the point with the maximum membership degree of the consequent T1 FS of the kth rule. Though the rulebase looks simple, it actually represents the most frequently used T1 FLS in practice.

TABLE I
EXAMPLE RULEBASE OF A T1 FLS WITH TWO INPUTS AND THREE MFS FOR EACH INPUT

$x_1 \setminus x_2$	X_{21}	X_{22}	X_{23}
X_{11}	y_1	y_2	y_3
X_{12}	y_4	y_5	y_6
X_{13}	y_7	y_8	y_9

For an input $\mathbf{x} = (x_1, x_2, \ldots, x_M)$, the output of a T1 FLS with the aforesaid structure is computed as

$$y(\mathbf{x}) = \frac{\sum_{k=1}^{K} f_k y_k}{\sum_{k=1}^{K} f_k} \tag{1}$$

where f_k is the firing level of $\mathbf{x}$ for the kth rule, which is computed by a t-norm, i.e.,

$$f_k = \mu_{X_{1,n_{1k}}}(x_1) \star \mu_{X_{2,n_{2k}}}(x_2) \star \cdots \star \mu_{X_{M,n_{Mk}}}(x_M). \tag{2}$$

Only minimum and product t-norms [25] are considered in this paper since they are the most frequently used ones in practice.

In this paper, we consider only continuous fuzzy sets (FSs) as MFs because discontinuous T1 FSs are almost never used in modeling and control.

Definition 5: A T1 FS X is c*ontinuous* if and only if its MF, $\mu_X(x)$, is a continuous function of x, i.e., for any c in its universe of discourse $\mathbb{X}$ and any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |\mu_X(x) - \mu_X(c)| < \epsilon$, where $\mu_X(x)$ is the membership grade of x on X.

C. Universal Approximators

Many authors have shown that various configurations of T1 FLSs are universal approximators [8], [9], [11], [27], [29], [30], [51], i.e., a T1 FLS can uniformly approximate any real continuous function on a compact domain to any degree of accuracy. For example, Wang and Mendel [51] proved that T1 FLSs with Gaussian MFs, product t-norm, and centroid defuzzification are universal approximators; Kreinovich *et al.* [30] further showed that such T1 FLSs are universal approximators for a smooth function, and also its derivatives, i.e., not only the smooth function is approximated by the T1 FLS, but also its derivatives; Castro [11] showed that T1 FLSs with Gaussian, triangular, or trapezoidal MFs, any t-norm, and any practical defuzzification method are universal approximators; Kosko [27] showed that all additive T1 FLSs[1] [26] are universal approximators. Kreinovich *et al.* [29] also gave a comprehensive review of many such results.

[1]Additive T1 FLSs [26] use summation (instead of maximum, as suggested by the extension principle [25], [67]) to combine the scaled consequent FSs and then use centroid defuzzification to obtain a crisp output.

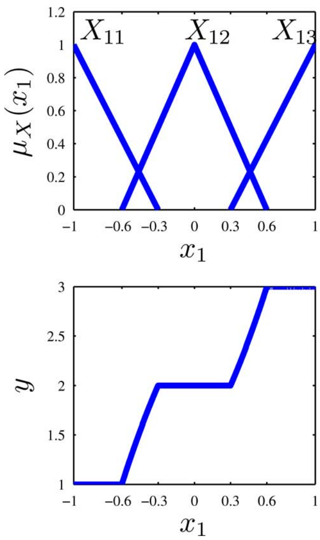

Fig. 1. Example input–output mappings of T1 FLSs with only one input.

TABLE II
RULEBASE FOR THE T1 FLSS SHOWN IN FIG. 1

x_1	X_{11}	X_{12}	X_{13}
y	1	2	3

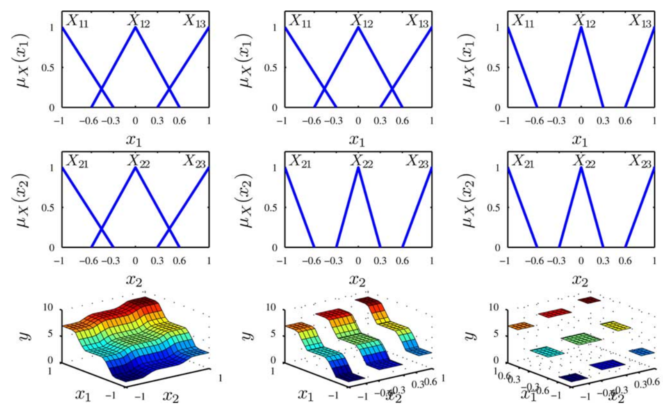

Fig. 2. Example input–output mappings of T1 FLSs with two inputs.

Intuitively, a T1 FLS must realize a continuous input–output mapping in order to approximate a continuous function to any degree of accuracy. This conjecture is mathematically proved in the following.

Theorem 1: A universal approximator $f(\mathbf{x})$ of a continuous function $g(\mathbf{x})$ must be continuous.

The proof of Theorem 1, in addition to proofs for all other theorems in this paper, are given in the Appendix.

So far, we have shown that as long as a T1 FLS is a universal approximator, it is continuous. According to Castro [11], T1 FLSs with Gaussian, triangular, or trapezoidal MFs, any t-norms, and centroid defuzzification are universal approximators, and hence, they are continuous. However, there are still two questions that remain unanswered.

1) Are T1 FLSs with arbitrary continuous MFs (not necessarily Gaussian, triangular, or trapezoidal) continuous?
2) In order to be a universal approximator, the T1 FSs must cover all input domains completely. What if there are gaps in at least one input domain?

These two questions are considered next.

D. Continuity of T1 FLSs

Consider the T1 FLS structure introduced in Section II-B.

Theorem 2: The T1 FLS $y(\mathbf{x})$ is *continuous* at $\mathbf{c} = (c_1, c_2, \ldots, c_M)$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{X_{mn}}(c_m) > 0$ for $\forall m = 1, 2, \ldots M$, i.e., every c_m is covered by some continuous T1 FSs.

Note that in this paper, "c_m is covered by some continuous T1 FSs" means that the membership grade of c_m on at least one of the T1 FSs is larger than 0, e.g., in the right column of Fig. 1, $x_1 = 0$ is covered by X_{12}, but $x_1 = \pm 0.3$ are not covered.

Theorem 3: The T1 FLS $y(\mathbf{x})$ has a *gap discontinuity* at $\mathbf{c} = (c_1, c_2, \ldots, c_M)$ if and only if there exists a c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{X_{mn}}(c_m) = 0$, i.e., there is at least one c_m not covered by any continuous T1 FS in its domain.

Observe that T1 FLSs cannot have jump discontinuities because Theorems 2 and 3 have covered all possible T1 FLSs.

For practical T1 FLSs, usually all inputs domains are fully covered by continuous T1 FSs, and hence, the T1 FLSs are continuous. Hence, people have not paid much attention to the continuity of T1 FLSs; however, the case is quite different and complicated for IT2 FLSs, as we will see in the next three sections.

E. Examples

Examples demonstrating Theorems 2 and 3 are presented in this section.

Fig. 1 shows two input–output mappings of T1 FLSs with only one input. The rulebase is shown in Table II, and product t-norm is used. The numbers in Table II are chosen only for illustration purpose. The first row of Fig. 1 shows the two MFs in the input domain and the second row the corresponding input–output mappings. Observe the following.

1) When the input MFs fully cover the input domain, as shown in the first column of Fig. 1, the corresponding input–output mapping is continuous, as indicated by Theorem 2.
2) When at least one point in the input domain is not covered by the MFs, the corresponding input–output mapping has gap discontinuities, as shown in the second column of Fig. 1. This result is consistent with Theorem 3.
3) The gaps in the output domain are determined by the uncovered intervals in the input domain, e.g., as shown in the second column of Fig. 1, x_1 has gaps at $x_1 = [-0.6, -0.3] \cup [0.3, 0.6]$, and hence, its input–output mapping also has gap discontinuities at $x_1 = [-0.6, -0.3] \cup [0.3, 0.6]$.

Fig. 2 shows the input–output mappings of three T1 FLSs with two inputs. The rulebase is shown in Table III, and product t-norm is used. Again, the numbers in Table III are chosen only for illustration purposes. From Fig. 2, observe the following.

TABLE III
RULEBASE FOR THE T1 FLSS SHOWN IN FIG. 2

$x_1 \setminus x_2$	X_{21}	X_{22}	X_{23}
X_{11}	1	2	3
X_{12}	4	5	6
X_{13}	7	8	9

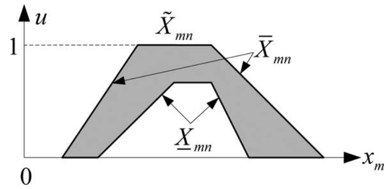

Fig. 3. IT2 FS $\tilde{X}_{mn}$ and its UMF $\overline{X}_{mn}$ and LMF $\underline{X}_{mn}$. Shaded area is the footprint of uncertainty (FOU).

1) When the input MFs fully cover the input domains, as shown in the first column of Fig. 2, the corresponding input–output mapping is continuous, as indicated by Theorem 2.
2) When at least one point in the input domain is not covered by the MFs, the corresponding input–output mapping has gap discontinuities, as shown in the last two columns of Fig. 2. These results are consistent with Theorem 3.
3) The gaps in the output domain are determined by the uncovered intervals in the input domains, e.g., in the second column of Fig. 2, x_2 is uncovered at $[-0.6, -0.3] \cup [0.3, 0.6]$, and hence, the input–output mapping is discontinuous at $x_2 \in [-0.6, -0.3] \cup [0.3, 0.6]$, as indicated by Theorem 3. Similarly, as shown in the third column of Fig. 2, both x_1 and x_2 are uncovered at $[-0.6, -0.3] \cup [0.3, 0.6]$, and hence, the input–output mapping has gap discontinuities at $[-0.6, -0.3] \cup [0.3, 0.6]$ in both the x_1 and x_2 domains.

III. CONTINUITY OF-IT2 FUZZY LOGIC SYSTEMS: KARNIK–MENDEL TYPE-REDUCTION AND CENTER-OF-SETS DEFUZZIFICATION

Karnik–Mendel (KM) type-reduction and center-of-sets defuzzification [23], [42], [58] are so far the most popular type-reduction and defuzzification methods for IT2 FLSs. The continuity of such IT2 FLSs is studied in this section. IT2 FLSs using five other type-reduction and defuzzification methods are investigated in the next two sections.

A. Structure of the IT2 FLS

Again, we consider only multiantecedent single-consequent IT2 FLSs in this section; however, our results can be easily extended to multiantecedent multiconsequent IT2 FLSs, because the latter can be decomposed into several multiantecedent single-consequent IT2 FLSs.

An example IT2 FS $\tilde{X}_{mn}$ is shown in Fig. 3. Its upper membership function (UMF) is denoted $\overline{X}_{mn}$, and its lower membership function (LMF) is denoted $\underline{X}_{mn}$.

The IT2 FLS has M inputs $\{x_m\}_{m=1,2,\ldots,M}$ and one output y. Assume the mth input has N_m MFs in its universe of discourse $\mathbb{X}_m$. Denote the nth MF in the mth input domain as $\tilde{X}_{mn}$. A complete rulebase with all possible combinations of the input MFs consists of $K = \prod_{m=1}^{M} N_m$ rules in the form of

TABLE IV
EXAMPLE RULEBASE OF AN IT2 FLS WITH TWO INPUTS AND THREE MFS FOR EACH INPUT

$x_1 \setminus x_2$	$\tilde{X}_{21}$	$\tilde{X}_{22}$	$\tilde{X}_{23}$
$\tilde{X}_{11}$	$[\underline{y}_1, \overline{y}_1]$	$[\underline{y}_4, \overline{y}_4]$	$[\underline{y}_7, \overline{y}_7]$
$\tilde{X}_{12}$	$[\underline{y}_2, \overline{y}_2]$	$[\underline{y}_5, \overline{y}_5]$	$[\underline{y}_8, \overline{y}_8]$
$\tilde{X}_{13}$	$[\underline{y}_3, \overline{y}_3]$	$[\underline{y}_6, \overline{y}_6]$	$[\underline{y}_9, \overline{y}_9]$

$\tilde{R}^k$: IF x_1 is $\tilde{X}_{1,n_{1k}}$ and ...and x_M is $\tilde{X}_{M,n_{Mk}}$, THEN y is $[\underline{y}_k, \overline{y}_k]$, $n_{ik} = 1, 2, \ldots, N_i$, $k = 1, 2, \ldots, K$

where $[\underline{y}_k, \overline{y}_k]$ is a constant interval, and generally, it is different for different rules. An example rulebase for an IT2 FLS with two inputs ($M = 2$) and three MFs for each input ($N_1 = N_2 = 3$) is shown in Table IV. Note that this IT2 FLS can be viewed as a Mamdani model with center-of-sets type-reduction and centroid defuzzification [42], i.e., $[\underline{y}_k, \overline{y}_k]$ represents the centroid of the consequent IT2 FS of the kth rule. When $\underline{y}_k = \overline{y}_k$, this rulebase represents the simplest TSK model, where each rule consequent is represented by a crisp number. Again, this rulebase represents the most commonly used IT2 FLSs in practice.

When KM type-reduction and center-of-sets defuzzification are used, the output of an IT2 FLS with the aforesaid structure for an input $\mathbf{x} = (x_1, x_2, \ldots, x_M)$ is computed as [42]

$$y(\mathbf{x}) = \frac{y_l(\mathbf{x}) + y_r(\mathbf{x})}{2} \tag{3}$$

where

$$y_l(\mathbf{x}) = \min_{\forall f_k \in [\underline{f}_k, \overline{f}_k]} \frac{\sum_{k=1}^{K} f_k \underline{y}_k}{\sum_{k=1}^{K} f_k} \tag{4}$$

$$= \frac{\sum_{k=1}^{k_l} \overline{f}_k \underline{y}_k + \sum_{k=k_l+1}^{K} \underline{f}_k \underline{y}_k}{\sum_{k=1}^{k_l} \overline{f}_k + \sum_{k=k_l+1}^{K} \underline{f}_k} \tag{5}$$

$$y_r(\mathbf{x}) = \max_{\forall f_k \in [\underline{f}_k, \overline{f}_k]} \frac{\sum_{k=1}^{K} f_k \underline{y}_k}{\sum_{k=1}^{K} f_k} \tag{6}$$

$$= \frac{\sum_{k=1}^{k_r} \underline{f}_k \overline{y}_k + \sum_{k=k_r+1}^{K} \overline{f}_k \overline{y}_k}{\sum_{k=1}^{k_r} \underline{f}_k + \sum_{k=k_r+1}^{K} \overline{f}_k} \tag{7}$$

in which $[\underline{f}_k, \overline{f}_k]$ is the firing interval of the kth rule, i.e.

$$\underline{f}_k = \mu_{\underline{X}_{1,n_{1k}}}(x_1) \star \mu_{\underline{X}_{2,n_{2k}}}(x_2) \star \cdots \star \mu_{\underline{X}_{M,n_{Mk}}}(x_M) \tag{8}$$

$$\overline{f}_k = \mu_{\overline{X}_{1,n_{1k}}}(x_1) \star \mu_{\overline{X}_{2,n_{2k}}}(x_2) \star \cdots \star \mu_{\overline{X}_{M,n_{Mk}}}(x_M). \tag{9}$$

Observe that both $\underline{f}_k$ and $\overline{f}_k$ are continuous functions when all IT2 MFs are continuous. Note also that $\{\underline{y}_k\}$ and $\{\overline{y}_k\}$ have been sorted in ascending order in (5) and (7), respectively. The *switch points* k_l and k_r are determined by the KM algorithms [23], [42] or the Enhanced KM (EKM) algorithms [56]–[58], and they satisfy

$$\underline{y}_{k_l} \le y_l(\mathbf{x}) \le \underline{y}_{k_l+1} \tag{10}$$

$$\overline{y}_{k_r} \le y_r(\mathbf{x}) \le \overline{y}_{k_r+1}. \tag{11}$$

Only continuous IT2 FSs are of interest in this paper, which are defined as follows:

Definition 6: An IT2 FS $\tilde{X}$ is *continuous* if and only if both its UMF and its LMF are continuous T1 FSs.

The continuity of the IT2 FLS is more interesting and complicated than the T1 FLS because, unlike the T1 FLS introduced in Section II-B, the output of the IT2 FLS does not have a closed-form solution. Furthermore, the KM algorithms for type-reduction involve switch points, which give the impression of discontinuity.

B. Continuity of IT2 FLSs

Two theorems on the discontinuities of IT2 FLSs are introduced next.

Theorem 4: The IT2 FLS has a *gap discontinuity* at $\mathbf{c} = \{c_1, c_2, \ldots, c_M\}$ if and only if $\exists c_m$ such that $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$, i.e., there exist at least one c_m not covered by the UMFs.

Theorem 5: The IT2 FLS has a *jump discontinuity* at $\mathbf{c} = \{c_1, c_2, \ldots, c_M\}$ if and only if we have the following.

1) $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(x_m) > 0$ for $\forall x_m$, i.e., each input domain is fully covered by the UMFs; and,
2) $\exists c_m$ such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, i.e., there exists at least one c_m not covered by the LMFs; and,
3) There exists an $m' \neq m$ such that, the minimum $\underline{y}_k$ and/or maximum $\overline{y}_k$ for all fired rules (i.e., those rules with $\overline{f}_k > 0$) changes as $c_{m'}$ changes to $c_{m'} + \delta$, where δ is an arbitrarily small positive or negative number.

The following corollary can be easily derived from Theorem 5:

Corollary 1: The IT2 FLS has a *jump discontinuity* at $\mathbf{c} = \{c_1, c_2, \ldots, c_M\}$ if we have the following.

1) The input domain is fully covered by the UMFs; and,
2) There exists at least one c_m not covered by the LMFs; and,
3) All rules have different consequents.

The third criterion in Theorem 5 requires $m' \neq m$, i.e., there must be at least two inputs in order to have jump discontinuities. Hence, we have the following:

Corollary 2: An IT2 FLS with only one input does not have *jump discontinuities*.

Theorems 4 and 5 suggest that an IT2 FLS can have both gap and jump discontinuities, whereas a T1 FLS can only have gap discontinuities.

By finding the complement of Theorems 4 and 5, we have the following necessary and sufficient conditions of a continuous IT2 FLS:

Theorem 6: The IT2 FLS is *continuous* at $\mathbf{c} = \{c_1, c_2, \ldots, c_M\}$ if and only if we have the following.

1) $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall m$; or,
2) For every m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, the minimum $\underline{y}_k$ and maximum $\overline{y}_k$ of all fired rules do not change as any $c_{m'}$ $(m' \neq m)$ changes to $c_{m'} + \delta$, where δ is an arbitrarily small positive or negative number.

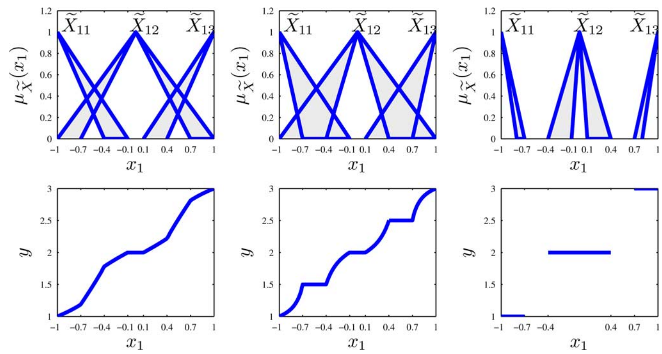

Fig. 4. Example input–output mappings of IT2 FLSs with only one input.

TABLE V
RULEBASE FOR THE IT2 FLSS SHOWN IN FIG. 4

x_1	$\tilde{X}_{11}$	$\tilde{X}_{12}$	$\tilde{X}_{13}$
y	$[0.8, 1.2]$	$[1.8, 2.2]$	$[2.8, 3.2]$

Since the second condition of Theorem 6 is more difficult to test than the first one, we suggest that practitioners who want to avoid both gap and jump discontinuities should focus on satisfying the first condition. Essentially, the first condition of Theorem 6 says that *an IT2 FLS is continuous as long as its input domain is fully covered by* both *the UMFs and the LMFs*. It is not a tight constraint on the shapes of the IT2 FS MFs, e.g., it is satisfied by all IT2 FLSs in [2], [20], [44], [60], [62], [63]; therefore, it should not limit the modeling power of IT2 FLSs.

C. Examples

Examples demonstrating Theorems 4–6 are presented in this section.

Fig. 4 shows three input–output mappings of IT2 FLSs with only one input. The first row shows the three MFs in the input domain and the second row the corresponding input–output mappings. The corresponding rulebase is given in Table V. Observe from Fig. 4 that we have the following.

1) When both the input UMFs and the input LMFs fully cover the input domain, as shown in the first column of Fig. 4, the corresponding input–output mapping is continuous, as indicated by Theorem 6.
2) When the input domain is fully covered by the UMFs but at least one point in the input domain is not covered by the LMFs, as shown in the middle column of Fig. 4, the corresponding input–output mapping is still continuous for the 1-input case, because it does not satisfy the third criterion of Theorem 5.
3) When the input UMFs do not fully cover the input domain, as shown in the last column of Fig. 4, the corresponding input–output mapping has gap discontinuities, as indicated by Theorem 4.

Fig. 4 demonstrates that an IT2 FLS with only one input cannot have jump discontinuities, as suggested by Corollary 2; however, that IT2 FLS can still have gap discontinuities.

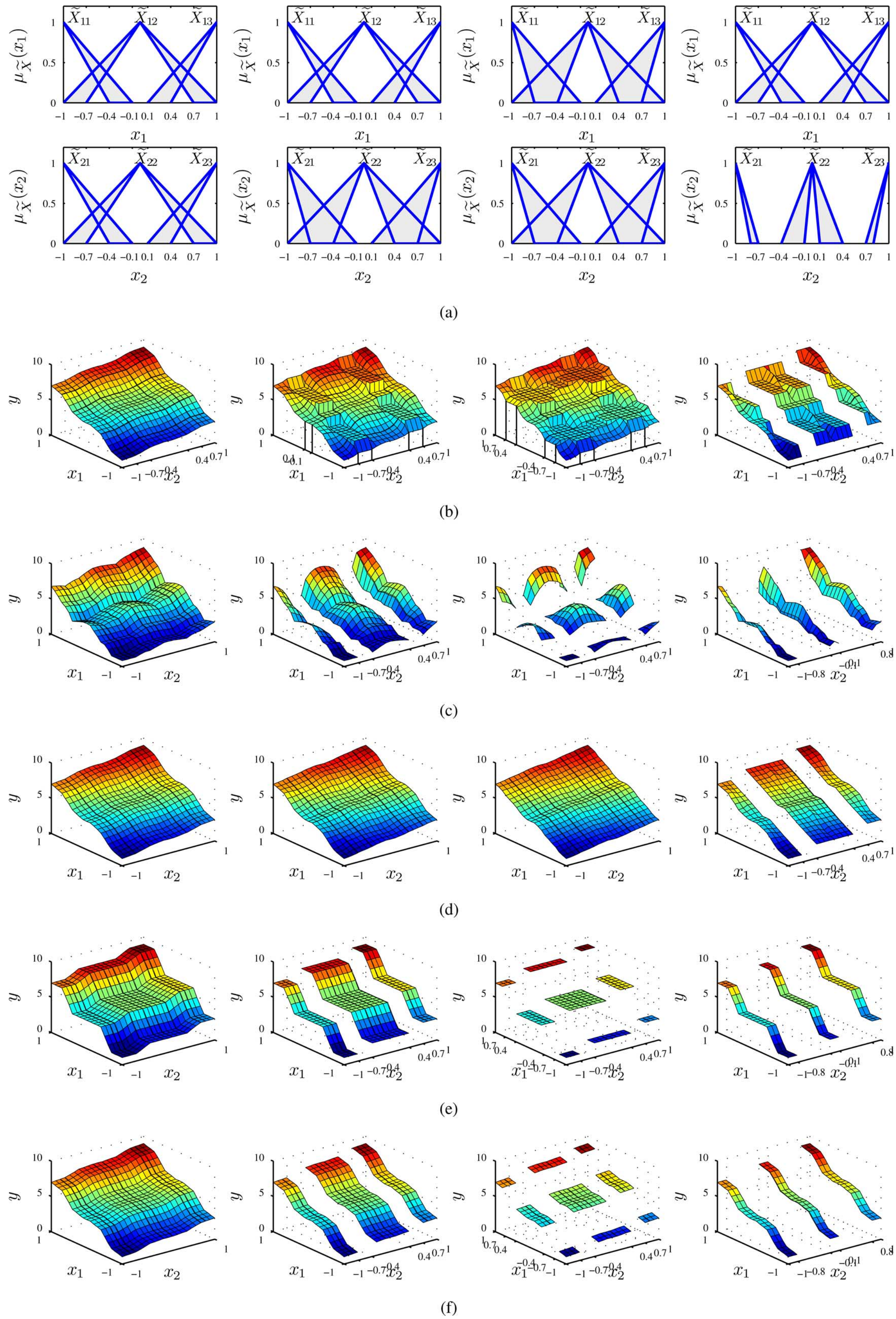

Fig. 5. Example input–output mappings of IT2 FLSs with two inputs. (a) Input MFs. (b) KM method. (c) UB method. (d) WT method ($\alpha_{mn} = 0.5$) and the NT method. (e) DY method. (f) BMM method.

TABLE VI
RULEBASE FOR THE IT2 FLSS SHOWN IN FIG. 5

$x_1 \setminus x_2$	$\tilde{X}_{21}$	$\tilde{X}_{22}$	$\tilde{X}_{23}$
$\tilde{X}_{11}$	$[0.8, 1.2]$	$[1.8, 2.2]$	$[2.8, 3.2]$
$\tilde{X}_{12}$	$[3.8, 4.2]$	$[4.8, 5.2]$	$[5.8, 6.2]$
$\tilde{X}_{13}$	$[6.8, 7.2]$	$[7.8, 8.2]$	$[8.8, 9.2]$

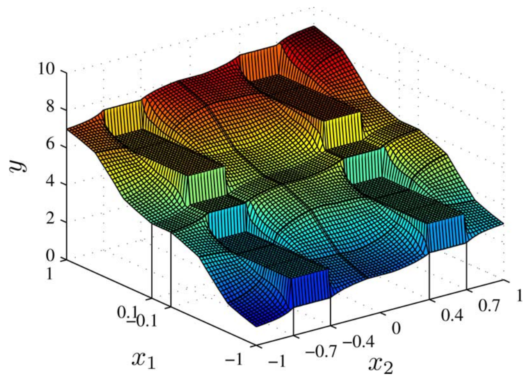

Fig. 6. Detailed illustration of jump discontinuities. The input MFs are shown in the second column of Fig. 5(a).

The following example illustrates the input–output mappings of four IT2 FLSs with two inputs. Fig. 5(a) shows the three MFs in each input domain, and Fig. 5(b) shows the corresponding input–output mappings using the KM type-reducer. The rulebase is given in Table VI. Observe from Fig. 5(b) that we have the following.

1) When both the input UMFs and the input LMFs fully cover the input domain, as shown in the first column of Fig. 5(a), the corresponding input–output mapping is continuous, as indicated by Theorem 6.
2) When the input domain is fully covered by the UMFs but at least one point in the input domain is not covered by the LMFs, as shown in the middle two columns of Fig. 5(a), the corresponding input–output mapping has jump discontinuities [e.g., when $x_1 = \pm 0.1$ and $x_2 \in [0.4, 0.7]$ in the second column of Fig. 5(a)], as indicated by Theorem 5. Observe also from the second column of Fig. 5(a) that even though it is the x_2 domain that is not fully covered by the LMFs, the jump discontinuities happen in the domain of x_1.
3) When the input UMFs do not fully cover the input domain, as shown in the last column of Fig. 5(a), the corresponding input–output mapping has gap discontinuities, as indicated by Theorem 4. Note that the x_2 domain is not covered by the UMFs; therefore, so the gap discontinuities happen in the x_2 domain, which is also indicated by Theorem 4.

Theorems 4 is intuitive. Next, the second column of Fig. 5(a) is used as an example to explain in detail how the jump discontinuities suggested by Theorem 5 are generated. A more detailed plot of the input–output mapping shown in the second column of Fig. 5(b) is depicted in Fig. 6. Observe that when $x_1 = \pm 0.1$, there are jump discontinuities at $x_2 \in [-0.7, -0.4] \cup [0.4, 0.7]$. The reason is analyzed next.

Observe from the second column of Fig. 5(a) that $\max_{n=1,2,3} \mu_{\overline{X}_{1n}}(x_1) > 0$ and $\max_{n=1,2,3} \mu_{\overline{X}_{2n}}(x_2) > 0$, i.e., the first criterion in Theorem 5 is satisfied. Consider $x_2 = 0.4$, where $\max_{n=1,2,3} \mu_{\underline{X}_{2n}}(x_2) = 0$, i.e., the second criterion of Theorem 5 is also satisfied. Further, consider the case that x_1 changes from 0.1 to $0.1 + \delta$, where $\delta > 0$ is an arbitrarily small number. Then, we have the following.

1) When $x_1 = 0.1$ and $x_2 = 0.4$, the firing intervals of the antecedents and rules are given in Table VII. Observe that only the two rules with antecedents $(\tilde{X}_{12}, \tilde{X}_{22})$ and $(\tilde{X}_{12}, \tilde{X}_{23})$ are fired, and the lower bounds of both firing intervals are 0; hence, from the KM algorithms

$$y_l(\mathbf{x}) = \frac{.54 \times 4.8 + 0 \times 5.8}{.54 + 0} = 4.8 \tag{12}$$

$$y_r(\mathbf{x}) = \frac{0 \times 5.2 + .3 \times 6.2}{0 + .3} = 6.2 \tag{13}$$

$$y(\mathbf{x}) = \frac{4.8 + 6.2}{2} = 5.5. \tag{14}$$

2) When $x_1 = 0.1 + \delta$ and $x_2 = 0.4$, the firing intervals of the antecedents and rules are given in Table VIII. Observe that only four rules are fired, and the lower bounds of all firing intervals are again 0. Observe also that the minimum $\underline{y}_k$ for all fired rules is 4.8 in both Tables VII and VIII; however, the maximum $\overline{y}_k$ for all fired rules changes from 6.2 to 9.2. Therefore, according to Theorem 5, there must be a jump discontinuity. Indeed

$$\begin{aligned} y_l(\mathbf{x}) &= \frac{(.54 - .6\delta) \times 4.8 + 0 \times 5.8 + 0 \times 7.8 + 0 \times 8.8}{.54 - .6\delta + 0 + 0 + 0} \\ &= 4.8 \end{aligned} \tag{15}$$

$$\begin{aligned} y_r(\mathbf{x}) &= \frac{0 \times 5.2 + 0 \times 6.2 + 0 \times 8.2 + \delta/2.7 \times 9.2}{0 + 0 + 0 + \delta/2.7} \\ &= 9.2 \end{aligned} \tag{16}$$

$$y(\mathbf{x}) = \frac{4.8 + 9.2}{2} = 7. \tag{17}$$

Therefore, $y(\mathbf{x})$ jumps from 5.5 to 7 when $x_2 = 0.4$ and x_1 moves from 0.1 to $0.1 + \delta$. Other jump discontinuities can be analyzed in a similar way.

IV. CONTINUITY OF IT2 FLSS: UNCERTAINTY BOUND TYPE-REDUCTION AND CENTER-OF-SETS DEFUZZIFICATION

The uncertainty bound (UB) type-reducer, proposed by Wu and Mendel [64], has also been widely used in fuzzy modeling and control [20], [38], [39]. The continuity of IT2 FLS using UB type-reduction and center-of-sets defuzzification is studied in this section.

Using the structure and notations introduced in Section III-A, the output of an IT2 FLS using UB type-reduction and center-of-sets defuzzification is again computed by (3), but now

$$y_l(\mathbf{x}) = \frac{\underline{y}_l(\mathbf{x}) + \overline{y}_l(\mathbf{x})}{2} \tag{18}$$

TABLE VII
FIRING INTERVALS OF THE IT2 FLSS SHOWN IN THE MIDDLE COLUMN OF FIG. 5 WHEN $x_1 = 0.1$ AND $x_2 = 0.4$

Firing interval of antecedents		Fired rules	
x_1 domain	x_2 domain	Firing interval	Rule consequent
$[\mu_{\underline{X}_{11}}(0.1), \mu_{\overline{X}_{11}}(0.1)] = [0,0]$	$[\mu_{\underline{X}_{21}}(0.4), \mu_{\overline{X}_{11}}(0.4)] = [0,0]$	$[6/7, 9/10] \times [0, 3/5] = [0, .54]$	$[4.8, 5.2]$
$[\mu_{\underline{X}_{12}}(0.1), \mu_{\overline{X}_{12}}(0.1)] = [6/7, 9/10]$	$[\mu_{\underline{X}_{22}}(0.4), \mu_{\overline{X}_{12}}(0.4)] = [0, 3/5]$	$[6/7, 9/10] \times [0, 1/3] = [0, .3]$	$[5.8, 6.2]$
$[\mu_{\underline{X}_{13}}(0.1), \mu_{\overline{X}_{13}}(0.1)] = [0,0]$	$[\mu_{\underline{X}_{23}}(0.4), \mu_{\overline{X}_{13}}(0.4)] = [0, 1/3]$		

TABLE VIII
FIRING INTERVALS OF THE IT2 FLSS SHOWN IN THE MIDDLE COLUMN OF FIG. 5 WHEN $x_1 = 0.1 + \delta$ AND $x_2 = 0.4$

Firing interval of antecedents		Fired rules	
x_1 domain	x_2 domain	Firing interval	Rule consequent
$[\mu_{\underline{X}_{11}}(0.1+\delta), \mu_{\overline{X}_{11}}(0.1+\delta)] = [0,0]$	$[\mu_{\underline{X}_{21}}(0.4), \mu_{\overline{X}_{11}}(0.4)] = [0,0]$	$[0, .54 - .6\delta]$	$[4.8, 5.2]$
$[\mu_{\underline{X}_{12}}(0.1+\delta), \mu_{\overline{X}_{12}}(0.1+\delta)] = [(.6-\delta)/.7, .9-\delta]$	$[\mu_{\underline{X}_{22}}(0.4), \mu_{\overline{X}_{12}}(0.4)] = [0, 3/5]$	$[0, .3 - \delta/3]$	$[5.8, 6.2]$
$[\mu_{\underline{X}_{13}}(0.1+\delta), \mu_{\overline{X}_{13}}(0.1+\delta)] = [0, \delta/.9]$	$[\mu_{\underline{X}_{23}}(0.4), \mu_{\overline{X}_{13}}(0.4)] = [0, 1/3]$	$[0, \delta/1.5]$	$[7.8, 8.2]$
		$[0, \delta/2.7]$	$[8.8, 9.2]$

$$y_r(\mathbf{x}) = \frac{\underline{y}_r(\mathbf{x}) + \overline{y}_r(\mathbf{x})}{2} \tag{19}$$

where

$$\overline{y}_l(\mathbf{x}) = \min\{\underline{y}^{(0)}(\mathbf{x}), \underline{y}^{(K)}(\mathbf{x})\} \tag{20}$$

$$\underline{y}_r(\mathbf{x}) = \max\{\overline{y}^{(0)}(\mathbf{x}), \overline{y}^{(K)}(\mathbf{x})\} \tag{21}$$

$$\underline{y}_l(\mathbf{x}) = \overline{y}_l(\mathbf{x}) - \frac{\sum_{k=1}^K(\overline{f}_k - \underline{f}_k)}{\sum_{k=1}^K \overline{f}_k \sum_{k=1}^K \underline{f}_k} \cdot \frac{\sum_{k=1}^K \underline{f}_k(\underline{y}_k - \underline{y}_1)\sum_{k=1}^K \overline{f}_k(\underline{y}_K - \underline{y}_k)}{\sum_{k=1}^K \underline{f}_k(\underline{y}_k - \underline{y}^1) + \sum_{k=1}^K \overline{f}_k(\underline{y}_K - \underline{y}_k)} \tag{22}$$

$$\overline{y}_r(\mathbf{x}) = \underline{y}_r(\mathbf{x}) + \frac{\sum_{k=1}^K(\overline{f}_k - \underline{f}_k)}{\sum_{k=1}^K \overline{f}_k \sum_{k=1}^K \underline{f}_k} \cdot \frac{\sum_{k=1}^K \overline{f}_k(\overline{y}_k - \overline{y}_1)\sum_{k=1}^K \underline{f}_k(\overline{y}_K - \overline{y}_k)}{\sum_{k=1}^K \overline{f}_k(\overline{y}_k - \overline{y}^1) + \sum_{k=1}^K \underline{f}_k(\overline{y}_K - \overline{y}_k)} \tag{23}$$

in which

$$\underline{y}^{(0)}(\mathbf{x}) = \frac{\sum_{k=1}^K \underline{f}_k \underline{y}_k}{\sum_{k=1}^K \underline{f}_k} \tag{24}$$

$$\underline{y}^{(K)}(\mathbf{x}) = \frac{\sum_{k=1}^K \overline{f}_k \underline{y}_k}{\sum_{k=1}^K \overline{f}_k} \tag{25}$$

$$\overline{y}^{(K)}(\mathbf{x}) = \frac{\sum_{k=1}^K \underline{f}_k \overline{y}_k}{\sum_{k=1}^K \underline{f}_k} \tag{26}$$

$$\overline{y}^{(0)}(\mathbf{x}) = \frac{\sum_{k=1}^K \overline{f}_k \overline{y}_k}{\sum_{k=1}^K \overline{f}_k}. \tag{27}$$

Theorem 7: The IT2 FLS using UB type-reduction and center-of-sets defuzzification is *continuous* at $\mathbf{c}$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$, i.e., every c_m is covered by some LMFs.

Theorem 8: The IT2 FLS using UB type-reduction and center-of-sets defuzzification has a *gap discontinuity* at $\mathbf{c}$ if and only if there exists c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$.

Observe from Theorems 7 and 8 that an IT2 FLS using UB type-reduction and center-of-sets defuzzification cannot have jump discontinuities because the conditions in Theorems 7 and 8 have covered all possible such IT2 FLSs.

Examples illustrating Theorems 7 and 8 are shown in Fig. 5(c). Observe that the UB method more easily results in gap discontinuities than the KM method. Additionally, observe from the last columns of Fig. 5(b) and (c) that the gaps resulting from the UB method are larger than those from the KM method because the former are determined by the gaps of the LMFs, whereas the latter are determined by the gaps of the UMFs.

V. CONTINUITY OF IT2 FUZZY LOGIC SYSTEMS WITH OTHER TYPE-REDUCTION AND DEFUZZIFICATION METHODS

In this section, four other methods,[2] which combine type-reduction and defuzzification, are introduced, and the continuities of the corresponding IT2 FLSs are investigated. All four of these methods require $\underline{y}_k = \overline{y}_k \equiv y_k$ for $\forall k = 1, 2, \ldots, K$.

A. Wu–Tan (WT) Method

Wu and Tan [60] proposed a closed-form type-reduction and defuzzification method by making use of the equivalent T1 FSs [61]. The basic idea is to first find an equivalent T1 membership grade $\mu_{X_{mn}}(x_m)$ to replace each firing interval $[\mu_{\underline{X}_{mn}}(x_m), \mu_{\overline{X}_{mn}}(x_m)]$, i.e.,

$$\mu_{X_{mn}}(x_m) = \mu_{\overline{X}_{mn}}(x_m) - h_{mn}(\mathbf{x})[\mu_{\overline{X}_{mn}}(x_m) - \mu_{\underline{X}_{mn}}(x_m)] \tag{28}$$

where $h_{mn}(\mathbf{x})$ is a function of the inputs and is different for different IT2 FSs. Then, the firing strengths of the rules become point (instead of interval) numbers computed from these $\mu_{X_{mn}}(x_m)$, and the output of the IT2 FLS is then computed as

$$y(\mathbf{x}) = \frac{\sum_{k=1}^K f_k y_k}{\sum_{k=1}^K f_k}. \tag{29}$$

[2]There are several other methods [18], [19], [43], [45] which bypass type-reduction; however, they require the rule consequents to be IT2 FSs so that the union of the fired rule output sets can be computed and used, whereas our IT2 FLS structure defined in Section III-A only uses the centroids of the consequent IT2 FSs. Because our structure is simpler and much more widely used in fuzzy logic modeling and control, those methods proposed in [18], [19], [43], [45] are not considered in this paper.

Theorem 9: The IT2 FLS computed by the WT method is *continuous* at $\mathbf{c}$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0$ for $\forall c_m$, i.e., every c_m is covered by some UMFs.

Theorem 10: The IT2 FLS computed by the WT method has a *gap discontinuity* at $\mathbf{c}$ if and only if there exists c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$, i.e., at least one c_m is not covered by any UMF.

Note that an IT2 FLS computed by the WT method cannot have jump discontinuities because the two conditions considered in Theorems 9 and 10 have covered all possible such IT2 FLSs.

Examples illustrating Theorems 9 and 10 are shown in Fig. 5(d). $\alpha_{mn} = 0.5$ is used in all examples. Observe that the WT method is less easily to have discontinuities, and the input–output mappings are generally less complex than those obtained from the KM method.

B. Nie–Tan (NT) Method

Nie and Tan [44] proposed another closed-form type-reduction and defuzzification method called NT method, where the output of an IT2 FLS is computed as

$$y(\mathbf{x}) = \frac{\sum_{k=1}^{K} (\underline{f}_k + \overline{f}_k) y_k}{\sum_{k=1}^{K} (\underline{f}_k + \overline{f}_k)}. \tag{30}$$

Observe that the NT method is a special case of the WT method when $h_{mn}(\mathbf{x}) = 0.5$. Therefore, it has the same properties as the WT method. The following two theorems are, hence, given without proofs:

Theorem 11: The IT2 FLS computed by the NT method is *continuous* at $\mathbf{c}$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0$ for $\forall c_m$, i.e., every c_m is covered by some UMFs.

Theorem 12: The IT2 FLS computed by the NT method has a *gap discontinuity* at $\mathbf{c}$ if and only if there exist x_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$, i.e., at least one c_m is not covered by any UMF.

Note that an IT2 FLS computed by the NT method cannot have jump discontinuities because the two conditions considered in Theorems 11 and 12 have covered all possible such IT2 FLSs.

Examples illustrating Theorems 11 and 12 are shown in Fig. 5(d). Again, observe that the NT method is less easily to have discontinuities, and the input–output mappings are generally less complex than those obtained from the KM method.

C. Du–Ying (DY) Method

Du and Ying [13] proposed a closed-form type-reduction and defuzzification method, which is referred to as the Du–Ying (DY) method in this paper. It first computes the crisp outputs obtained by all possible combinations of the lower and upper firing levels, i.e.,

$$y_i(\mathbf{x}) = \frac{\sum_{k=1}^{K} f_k^* y_k}{\sum_{k=1}^{K} f_k^*}, \quad i = 1, 2, \ldots, 2^K \tag{31}$$

where $f_k^* \in \{\underline{f}_k, \overline{f}_k\}$. The final defuzzified output is then computed as the average of all these 2^K $y_i(\mathbf{x})$, i.e.,

$$y(\mathbf{x}) = \frac{1}{2^K} \sum_{i=1}^{2^K} y_i(\mathbf{x}). \tag{32}$$

Theorem 13: The IT2 FLS computed by the DY method is *continuous* at $\mathbf{c}$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$, i.e., every c_m is covered by some LMFs.

Theorem 14: The IT2 FLS computed by the DY method has a *gap discontinuity* at $\mathbf{c}$ if and only if there exist c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, i.e., at least one c_m is not covered by any LMF.

Note that an IT2 FLS computed by the DY method cannot have jump discontinuities, because the two conditions considered in Theorems 13 and 14 have covered all possible such IT2 FLSs.

Examples illustrating Theorems 13 and 14 are shown in Fig. 5(e). Observe that the DY method is more easily to have gap discontinuities than the KM method, and the gaps in the input–output mappings are larger than those from the KM method.

D. Begian–Melek–Mendel (BMM) Method

Begian *et al.* [2] proposed another closed-form type-reduction and defuzzification method for TSK IT2 FLSs, i.e.,

$$y(\mathbf{x}) = m \frac{\sum_{k=1}^{K} \underline{f}_k y_k}{\sum_{k=1}^{K} \underline{f}_k} + n \frac{\sum_{k=1}^{K} \overline{f}_k y_k}{\sum_{k=1}^{K} \overline{f}_k} \tag{33}$$

where m and n are adjustable coefficients.

Theorem 15: The IT2 FLS computed by the BMM method is *continuous* at $\mathbf{c}$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$, i.e., every c_m is covered by some LMFs.

Theorem 16: The IT2 FLS computed by the BMM method has a *gap discontinuity* at $\mathbf{c}$ if and only if there exist c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, i.e., at least one c_m is not covered by any LMF.

The proofs of Theorems 15 and 16 are very similar to those of Theorems 13 and 14, and therefore, they are left to the reader as an exercise.

Note that an IT2 FLS computed by the BMM method cannot have jump discontinuities because the two conditions considered in Theorems 15 and 16 have covered all possible such IT2 FLSs.

Examples illustrating Theorems 15 and 16 are shown in Fig. 5(f). Observe that the DY method is more easily to have gap discontinuities than the KM method, and the gaps in the input–output mappings are larger than those from the KM method.

VI. Summarization and Discussions

This section summarizes our results on the six type-reduction and defuzzification methods for IT2 FLSs and proposes some design guidelines and new research directions.

A. Summarization

Table IX summarizes the continuities of IT2 FLSs, constructed by the six different type-reduction and defuzzification

TABLE IX
SUMMARIZATION OF THE CONTINUITIES OF SIX TYPE-REDUCTION AND DEFUZZIFICATION METHODS AT $\mathbf{c}$

Method	$\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0\ \forall c_m$		$\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$
	$\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0\ \forall c_m$	$\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$	
KM	Continuous	Possible jump discontinuity	Gap discontinuity
UB	Continuous	Gap discontinuity	Gap discontinuity
WT	Continuous	Continuous	Gap discontinuity
NT	Continuous	Continuous	Gap discontinuity
DY	Continuous	Gap discontinuity	Gap discontinuity
BMM	Continuous	Gap discontinuity	Gap discontinuity

methods introduced in the previous three sections. Note that the table only compares the continuities of different type-reduction and defuzzification methods. It does not concern the performance of these methods. As a verification of our results, we find that no discontinuities of IT2 FLSs were observed in [2], [20], [44], [60], [62], [63] because all their IT2 FLSs satisfied the conditions that $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(x_m) > 0$ and $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(x_m) > 0$ for $\forall x_m$.

Summarizing the results in the first column of Table IX, and noting that Gaussian IT2 FSs always give $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(x_m) > 0$ and $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(x_m) > 0$ for $\forall x_m$, we have the following:

Theorem 17: All six type-reduction and defuzzification methods introduced in this paper always give *continuous* input–output mappings when Gaussian IT2 FSs are used.

Again, Theorem 17 only concerns the continuity of IT2 FLSs with Gaussian MFs; it does not imply that such IT2 FLSs would have better performance than those with triangular or trapezoidal MFs.

B. Guidelines for Designing Continuous IT2 FLSs

Based on all results introduced so far in this paper, particularly those summarized in Table IX, we have the following guidelines for practitioners who want to design continuous IT2 FLSs.

1) To guarantee a continuous input–output mapping regardless of which type-reduction and defuzzification method is used, Gaussian IT2 FSs should be employed.
2) When triangular and/or trapezoidal IT2 FSs are used, to guarantee a continuous input–output mapping, the LMFs should cover every input domain. This implies that the UMFs must also cover every input domain.

C. Smoothness and Monotonicity

As mentioned in Section I, continuity and smoothness are different concepts, i.e., smoothness requires that the derivatives of a function are also continuous. In this paper, we only considered the continuities of T1 and IT2 FLSs. One of our future research directions is to investigate their smoothness. To the authors' best knowledge, Kreinovich *et al.* [30] are the only ones to study the smoothness of T1 FLSs. They showed that T1 FLSs with Gaussian MFs, product t-norm, and centroid defuzzification are universal approximators for a smooth function, as well as its derivatives, i.e., not only the smooth function is approximated by the T1 FLS, but its derivatives as well. Using Theorem 1, this is equivalent to saying that the T1 FLS is smooth. To date, there have been no such results for IT2 FLSs.

Another interesting future direction is the monotonicity of FLSs. Many applications of fuzzy logic modeling and control require monotonicity of the output with respect to inputs, e.g., in queuing systems [28], *"the control actions concerning admission and routing of customers and allocation of resources depend monotonically on the queue sizes and the customer arrival rates [68],"* and a fuzzy logic controller for an air conditioner needs to increase the motor speed as the room temperature deviates more from the setpoint [69], etc. There have been several papers on the monotonicity of T1 FLSs [6], [7], [28], [36], [54], [55], [69]; however, there has been only one paper on the monotonicity of IT2 FLSs by Li *et al.* [32]. Additionally, Li *et al.* only considered single-input IT2 FLSs and only gave the sufficient conditions for monotonic IT2 FLSs using the KM type-reducer. To be more practical, more works need to be done on multi-input IT2 FLSs and IT2 FLSs using other type-reducers.

D. Applications to Hybrid and Switched Systems

So far, in this paper, we have emphasized continuous FLSs, because they are the most frequently used FLSs in practice; however, discontinuous FLSs may also be very useful in hybrid and switched systems [35], [48] modeling and control, which is becoming increasingly popular recently due to the wide applications of computers and digital controllers.

Hybrid systems [4] are finite-state machines coupled with controllers and plants modeled by differential or difference equations. They arise whenever logical decision making is mixed with the generation of continuous-valued control laws. Switched systems are an important class of hybrid systems. They consist of "*a finite number of continuous-time subsystems and a logical rule that orchestrates switching between them*" [48. p. 3]. Real-world examples of hybrid and switched systems include systems with relays, switches, and hysteresis [49], [53], computer disk drives [17], transmissions, stepper motors, and other motion controllers [5], biological applications [16], etc.

FLSs have been extensively used in hybrid and switched systems modeling and control [1], [15], [33], [37], [46], [47]; however, to the authors best knowledge, all these approaches consider each continuous-time subsystem separately. For example, consider a simple room temperature control problem: If the temperature is lower than $20\,^\circ\mathrm{C}$, then the heater is on; otherwise, the heater is off. Clearly, the differential equations representing the room temperature dynamics are different in the two states (with or without heater). The traditional modeling approach for switched systems would model the two discrete states separately; however, an FLS may be designed to have a discontinuity at 20 °C to model the two discrete states by a single FLS. This would simplify the model understanding and representation. Exactly how to do this is still under investigation.

VII. Conclusion

In this paper, the continuities and discontinuities of T1 and IT2 FLSs have been defined and investigated. Particularly, six different type-reduction and defuzzification methods for IT2 FLSs have been studied, which cover almost all practical IT2 FLSs. Conditions under which an FLS gives a continuous/discontinous input–output mapping were derived. Guidelines on designing continuous IT2 FLSs were also given. These results should be very useful in traditional fuzzy logic modeling and control, where usually, a continuous input–output mapping is desired, and in hybrid and switched systems modeling and control, where a discontinuous input–output mapping is needed. To date, we believe this to be the most comprehensive study on the continuity of T1 and IT2 FLSs. Our future research includes the study of smoothness and monotonicity of T1 and IT2 FLSs, as well as how to apply discontinuous FLSs to hybrid and switched systems modeling and control.

Appendix A
Proof of Theorem 1

Consider an arbitrary small number $\epsilon > 0$ and an arbitrary point $\mathbf{c} = (c_1, c_2, \dots, c_M)$ in the input domain of $f(\mathbf{x})$. Because $g(\mathbf{x})$ is a continuous function of $\mathbf{x}$, we can always find a $\delta_1 > 0$ such that when $\max_m |x_m - c_m| < \delta_1$, $|g(\mathbf{x}) - g(\mathbf{c})| < \epsilon/3$. Since $f(\mathbf{x})$ universally approximates $g(\mathbf{x})$, we always have $|g(\mathbf{c}) - f(\mathbf{c})| < \epsilon/3$ for $\forall\mathbf{c}$, and we can find $\delta_2 > 0$ such that $|g(\mathbf{x}) - f(\mathbf{c})| < \epsilon/3$ when $\max_m |x_m - c_m| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. Then, when $\max_m |x_m - c_m| < \delta$

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{c})| &= |f(\mathbf{x}) - f(\mathbf{c}) + g(\mathbf{x}) - g(\mathbf{x}) + g(\mathbf{c}) - g(\mathbf{c})| \\ &\leq |f(\mathbf{x}) - g(\mathbf{x})| + |g(\mathbf{c}) - f(\mathbf{c})| + |g(\mathbf{x}) - g(\mathbf{c})| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned} \tag{34}$$

i.e., $f(\mathbf{x})$ is continuous at $\mathbf{c}$. Since $\mathbf{c}$ is an arbitrary point in the input domain of $f(\mathbf{x})$, $f(\mathbf{x})$ must be continuous in its entire input domain.

Appendix B
Proof of Theorem 2

We need to prove that $y(\mathbf{x})$ in (1) is continuous at $\mathbf{c} = (c_1, c_2, \dots, c_M)$. The firing level of the kth rule f_k is computed by (2). Since each $\mu_{X_{m,n_{mn}}}(x_m)$ is a continuous function of x_m, and both product and minimum t-norms are continuous functions, f_k must be continuous at $\mathbf{c}$. Consequently, $\sum_{k=1}^K f_k y_k$ and $\sum_{k=1}^K f_k$ are also continuous at $\mathbf{c}$. As a result, $y(\mathbf{x})$ in (1) is continuous at $\mathbf{c}$ if and only if $\sum_{k=1}^K f_k \neq 0$, which holds if and only if $\max_{n=1,2,\dots,N_m} \mu_{X_{mn}}(c_m) > 0$ for $\forall m = 1, 2, \dots M$, i.e., every c_m is covered by some continuous T1 FSs.

Appendix C
Proof of Theorem 3

Following the same line of reasoning in the proof of Theorem 2, $y(\mathbf{x})$ in (1) has a gap discontinuity at $\mathbf{c}$ if and only if $\sum_{k=1}^K f_k = 0$, which holds if and only if there exists c_m such that $\max_{n=1,2,\dots,N_m} \mu_{X_{mn}}(c_m) = 0$, i.e., c_m is not covered by any continuous T1 FS in its domain.

Appendix D
Proof of Theorem 4

To show $y(\mathbf{x})$ has a gap discontinuity at $\mathbf{c}$ is equivalent to showing that at least one of $y_l(\mathbf{x})$ and $y_r(\mathbf{x})$ has a gap discontinuity at $\mathbf{c}$, i.e., $y(\mathbf{x})$ is undefined as long as at least one of $y_l(\mathbf{x})$ and $y_r(\mathbf{x})$ is undefined. We will show that $y_l(\mathbf{x})$ has a gap discontinuity at $\mathbf{c}$ if and only if $\exists c_m$ such that $\max_{n=1,2,\dots,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$, i.e., there exist at least one c_m not covered by the UMFs. The condition is also true for $y_r(\mathbf{x})$. Since its proof is very similar to that for $y_l(\mathbf{x})$, it is left to the reader as an exercise.

Consider the sufficiency first. When $\exists c_m$ such that $\max_{n=1,2,\dots,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$, all $\underline{f}_k$ and $\overline{f}_k$, which is computed by (8) and (9), equal 0; hence, $\sum_{k=1}^K f_k$ in the numerator of (4) is always 0. Consequently, $y_l(\mathbf{x})$ is undefined at $\mathbf{c}$, i.e., $y_l(\mathbf{x})$ has a gap discontinuity at $\mathbf{c}$.

Next, consider the necessity. When $\max_{n=1,2,\dots,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0$ for $\forall m = 1, 2, \dots, M$, there are $K' \geq 1$ rules whose $\overline{f}_k > 0$, and $y_l(\mathbf{x})$ is computed by (35), which is defined in this case; therefore, $y_l(\mathbf{x})$ does not have a gap discontinuity at $\mathbf{c}$. Consequently, to have a gap discontinuity at $\mathbf{c}$, there must $\exists c_m$ such that $\max_{n=1,2,\dots,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$.

Appendix E
Proof of Theorem 5

A lemma on the sufficient condition of a continuous IT2 FLS is given first. It will be used in the proof of Theorem 5.

Lemma 1: The IT2 FLS is continuous at $\mathbf{c}$ if $\max_{n=1,2,\dots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall m = 1, 2, \dots M$, i.e., every c_m is covered by some continuous LMFs.

Proof: To prove $y(\mathbf{x})$ defined in (3) is continuous at $\mathbf{c}$, we need to show that both $y_l(\mathbf{x})$ and $y_r(\mathbf{x})$ are continuous at $\mathbf{c}$. We only show that for $y_l(\mathbf{x})$. The proof for $y_r(\mathbf{x})$ is very similar and, hence, is left to the reader as an exercise.

It has been shown in [14] and [41] that $y_l(\mathbf{x})$ can be written as

$$y_l(\mathbf{x}) = \min_{k' \in [1, K']} \frac{\sum_{k=1}^{k'} \overline{f}_k \underline{y}_k + \sum_{k=k'+1}^{K'} \underline{f}_k \underline{y}_k}{\sum_{k=1}^{k'} \overline{f}_k + \sum_{k=k'+1}^{K'} \underline{f}_k} \tag{35}$$

where K' is the number of fired rules, i.e., those rules with $\overline{f}_k > 0$. Therefore, to show $y_l(\mathbf{x})$ is continuous at $\mathbf{c}$, we only need to show each $(\sum_{k=1}^{k'} \overline{f}_k \underline{y}_k + \sum_{k=k'+1}^{K'} \underline{f}_k \underline{y}_k / \sum_{k=1}^{k'} \overline{f}_k + \sum_{k=k'+1}^{K'} \underline{f}_k)$ is continuous at $\mathbf{c}$, because the minimum of several continuous functions is still continuous. When $\max_{n=1,2,\dots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall m = 1, 2, \dots M$, i.e., every c_m is covered by some continuous LMFs, all $\underline{f}_k$ and $\overline{f}_k$

are continuous at $\mathbf{c}$, and hence, $\sum_{k=1}^{k'} \overline{f}_k + \sum_{k=k'+1}^{K'} \underline{f}_k > 0$; therefore, $y_l(\mathbf{x})$ is continuous at $\mathbf{c}$. □

Next, we prove Theorem 5. Consider the sufficiency first. When there exists an m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, all firing levels $\underline{f}_k$ are zero for $\mathbf{c}$. Consider $y_l(\mathbf{x})$ in (35). Its minimum is achieved when $k = 1$, i.e., $y_l(\mathbf{x}) = \underline{y}_1 = \min_{k=1,\ldots,K'} \underline{y}_k$.

When any $c_{m'}$ $(m' \neq m)$ changes to $c_{m'} + \delta$, because $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, all firing levels $\underline{f}_k$ are still zero; hence, $y_l(\mathbf{x})$ still equals the minimum $\underline{y}_k$ of all firing rules. Clearly, if the minimum $\underline{y}_k$ of all firing rules changes as any $c_{m'}$ $(m' \neq m)$ changes to $c_{m'} + \delta$, then there is a jump in $y_l(\mathbf{x})$, and hence, the input–output mapping has a jump discontinuity at $\mathbf{c}$. Similarly, if the maximum $\overline{y}_k$ of all firing rules changes as any $c_{m'}$ $(m' \neq m)$ changes to $c_{m'} + \delta$, then there is a jump in $y_r(\mathbf{x})$, and hence, the input–output mapping has a jump discontinuity at $\mathbf{c}$.

Next, consider the necessity that $y(\mathbf{c})$ can only have three cases.

1) There exists c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$.
2) $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0$ and $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for all c_m.
3) $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0$ for all c_m, but there exists c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$.

The first case has a gap discontinuity at $\mathbf{c}$, according to Theorem 4. The second case is continuous at $\mathbf{c}$, according to Lemma 1. Then, a jump discontinuity can only happen in the third case, as indicated by the first two criteria in Theorem 5. When there exists c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, the lower bounds of all fired rules are 0. In this case, $y_l(\mathbf{c})$ equals the minimum $\underline{y}_k$ of all fired rules, and $y_r(\mathbf{c})$ equals the maximum $\overline{y}_k$ of all fired rules. Therefore, if Criterion 3 of Theorem 5 is not satisfied, then both $y_l(\mathbf{c})$ and $y_r(\mathbf{c})$ are not changed when $\mathbf{c}$ changes; hence, there is no jump discontinuity at $\mathbf{c}$. In other words, to have a jump discontinuity at $\mathbf{c}$, the minimum $\underline{y}_k$ and/or maximum $\overline{y}_k$ of all fired rules must be different as $\mathbf{c}$ changes.

Appendix F
Proof of Theorem 7

Because $\mu_{\overline{X}_{mn}} \geq \mu_{\underline{X}_{mn}}$, $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$ implies that $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0$ for $\forall c_m$. To prove that the input–output mapping of an IT2 FLS is continuous at $\mathbf{c}$, we need to prove that both $y_l(\mathbf{x})$ and $y_r(\mathbf{x})$ are continuous at $\mathbf{c}$. The proof for $y_l(\mathbf{x})$ is given next. The proof for $y_r(\mathbf{x})$ is very similar and is left to the reader as an exercise.

Observe from (24) that $\underline{y}^{(0)}(\mathbf{x})$ is the output of a T1 FLS constructed from all the LMFs, and from (25) that $\underline{y}^{(K)}(\mathbf{x})$ is the output of a T1 FLS constructed from all the UMFs. Therefore, according to Theorem 2, both $\underline{y}^{(0)}(\mathbf{x})$ and $\underline{y}^{(K)}(\mathbf{x})$ are continuous at $\mathbf{c}$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$. Consequently, $\overline{y}_l(\mathbf{x})$ is continuous at $\mathbf{c}$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$. Because the bracketed term in (22) is also a continuous function of $\underline{f}_k$ and $\overline{f}_k$ (i.e., its derivatives with respect to $\underline{f}_k$ and $\overline{f}_k$ always exist) if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$, whereas $\underline{f}_k$ and $\overline{f}_k$ are continuous at $\mathbf{c}$, the bracketed term in (22) is also continuous at $\mathbf{c}$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$. Consequently, $\underline{y}_l(\mathbf{x})$ is continuous at $\mathbf{c}$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$. As $y_l(\mathbf{x})$ is a continuous function of $\underline{y}_l(\mathbf{x})$ and $\overline{y}_l(\mathbf{x})$, it is continuous at $\mathbf{c}$ if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$.

Appendix G
Proof of Theorem 8

Consider sufficiency first. When there exists c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, all $\underline{f}_k = 0$, and hence, $\underline{y}^{(0)}(\mathbf{x})$ is undefined, i.e., there is a gap discontinuity at $\mathbf{c}$.

The necessity of Theorem 8 can be easily seen from Theorem 7, i.e., when $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$, $y(\mathbf{x})$ is continuous at $\mathbf{c}$; therefore, to have a discontinuity, there must exist c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$.

Appendix H
Proof of Theorem 9

f_k are always continuous at $\mathbf{c}$ because all $\mu_{X_{mn}}(x_m)$ in (28) are continuous. Therefore, $y(\mathbf{x})$ is continuous at $\mathbf{c}$ if and only if $\sum_{k=1}^{K} f_k > 0$, which is true if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0$ for $\forall c_m$.

Appendix I
Proof of Theorem 10

$y(\mathbf{x})$ in (29) has a gap discontinuity at $\mathbf{c}$ if and only if it is undefined at $\mathbf{c}$, which happens if and only if $\sum_{k=1}^{K} f_k = 0$. $\sum_{k=1}^{K} f_k = 0$ happens if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$ for certain c_m. Theorem 10 is hence proved.

Appendix J
Proof of Theorem 13

$\underline{f}_k$ and $\overline{f}_k$ are continuous at $\mathbf{c}$ because all MFs are continuous. Therefore, all $y_i(\mathbf{x})$ are continuous at $\mathbf{c}$ if and only if $\sum_{k=1}^{K} \underline{f}_k > 0$ and $\sum_{k=1}^{K} \overline{f}_k > 0$, which are true if and only if $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall c_m$.

Appendix K
Proof of Theorem 14

$y(\mathbf{x})$ in (32) is undefined at $\mathbf{c}$ if and only if at least one $y_i(\mathbf{x})$ is undefined, which can happen if and only if [3] $\sum_{k=1}^{K} \underline{f}_k = 0$, which can happen if and only if $\underline{f}_k = 0$ for $\forall k = 1, 2, \ldots, K$. The latter can happen only for $\mathbf{c}$ that contains c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$.

[3] When $\sum_{k=1}^{K} \overline{f}_k = 0$, all $y_i(\mathbf{x})$ are undefined; however, $\sum_{k=1}^{K} \overline{f}_k = 0$ also implies $\sum_{k=1}^{K} \underline{f}_k = 0$ because $\underline{f}_k \leq \overline{f}_k$ for $\forall k = 1, 2, \ldots, K$.

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