# Interval Type-2 Fuzzy PI Controllers: Why They Are More Robust

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***Abstract*—Many experiments have shown that interval type-2 (IT2) fuzzy PI controllers are generally more robust than their type-1 (T1) counterparts, as they are better able to cope with disturbances and uncertainties and eliminate oscillations. This paper aims at providing theoretical explanations to these experimental observations. Analysis has shown that the additional degrees of freedom provided by the footprint of uncertainty enable an IT2 fuzzy PI controller to emulate a variable gain PI controller around the origin, and these PI gains are smaller than those of the baseline type-1 fuzzy PI controller; consequently, the same amount of disturbance will cause a smaller control signal change for an IT2 fuzzy PI controller, and hence reduces the risk of oscillation. The results in this paper enable us to connect traditional PI controllers with IT2 fuzzy PI controllers. Based on the closed-form equivalent PI gains, controller design rules in one domain may be transferred to the other domain, and it is possible to develop new control laws by combining the merits of both traditional PI controllers and IT2 fuzzy PI controllers.**

***Index Terms*—Interval type-2 fuzzy logic control, PI control, robust control**

## I. Introduction

Interval type-2 fuzzy logic controllers (IT2 FLCs) have been used in many applications [1], [2], [4], [6], [11], [15], [16]. Experiments showed that they have the potential to outperform their type-1 (T1) counterparts [1], [10], [15], [16], especially, they are better able to cope with disturbances and uncertainties and eliminate oscillations. One reason may be that each IT2 fuzzy set (FS) has an extra mathematical dimension due to its footprint of uncertainty (FOU), and hence with the same number of membership functions (MFs), an IT2 FLC offers more design freedom. However, experiments also showed that an IT2 FLC may achieve better performance with fewer design parameters than a T1 FLC [15], which cannot be explained using the number of design parameters. Our observation is that generally an IT2 FLC has a smoother control surface around the origin than a T1 FLC [15], consequently, the same amount of disturbance will cause a smaller control signal change, and hence reduces the risk of oscillation.

This paper aims at understanding the behavior of an IT2 FLC around the origin from a mathematical point of view. The most popular IT2 PI (Proportional-Integral) controllers are studied in this paper; however, the analysis can also be applied to IT2 PD (Proportional-Differential) and PID (Proportional-Integral-Differential) controllers. The analysis is performed by determining the equivalent PI gains around the origin as a function of the FOU size. By examining the equivalent PI gains, insights into why IT2 FLCs are better at handling modeling uncertainties are obtained.

The rest of the paper is organized as follows: Section II introduces the architecture of the IT2 fuzzy PI controller studied in this paper. Section III deduces the equivalent PI gains of the IT2 PI controller. Section IV verifies the equivalent PI gains by simulations. Finally, Section V draws conclusions.

## II. IT2 PI Controllers

A linear PI control law is usually implemented as

$$\dot{u} = K_P \dot{e} + K_I e \tag{1}$$

where $\dot{u}$ is the change of the control signal, $e$ is the feedback error, $\dot{e}$ is the change of error, and $K_P$ and $K_I$ are the Proportional and Integral gains, respectively. A T1 FLC with rulebase[1]

$$R^{ij}: \text{If } \dot{e} \text{ is } \dot{E}_i \text{ and } e \text{ is } E_j, \text{ then } \dot{u} \text{ is } \dot{u}_{ij}$$
$$i = -N, \ldots, N, j = -M, \ldots, M$$

implements the linear PI controller in (1) if [8]:

1) Triangular T1 FSs are used for input MFs $\dot{E}_i$ and $E_j$, and they are constructed in such a way that for any input the firing levels of all MFs add to 1; and,
2) The consequents of the rules are crisp numbers defined as

$$\dot{u}_{ij} = K_P \dot{e}_i + K_I e_j \tag{2}$$

where $\dot{e}_i$ and $e_j$ are apexes of the antecedent T1 MFs.

An example of such a T1 FLC is shown as the bold lines in Fig. 1.

An IT2 fuzzy PI controller can be constructed by blurring the T1 FSs to IT2 FSs, as shown in Fig. 1. The rulebase for the resulting IT2 FLC is

$$\tilde{R}^{ij}: \text{If } \dot{e} \text{ is } \widetilde{\dot{E}}_i \text{ and } e \text{ is } \widetilde{E}_j, \text{ then } \dot{u} \text{ is } \dot{u}_{ij}$$
$$i = -N, \ldots, N, j = -M, \ldots, M$$

[1]For control applications the input domains of $e$ and $\dot{e}$ are usually symmetrical about 0. The MFs are indexed symmetrically around 0 so that their positions with respect to 0 are obvious. Also, it is assumed that the number of "negative" MFs equals the number of "positive" MFs, which is a common practice for FLCs.

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where $\tilde{\dot{E}}_i$ and $\tilde{E}_j$ are IT2 FSs obtained by blurring $\dot{E}_i$ and $E_j$, respectively, and $\dot{u}_{ij}$ is the same as that defined in (2).

This paper aims at understanding the behavior of an IT2 FLC around the origin, and hence only the fuzzy partitions around the origin are of interest. These fuzzy partitions are shown in Fig. 1. For simplicity, the following assumptions are used:

1) The upper and lower MFs of $\tilde{\dot{E}}_{-1}$ and $\tilde{\dot{E}}_1$ are symmetrical about the baseline T1 MF $\dot{E}_{-1}$ and $\dot{E}_1$, respectively, as shown in Fig. 1(a).
2) The upper and lower MFs of $\tilde{E}_{-1}$ and $\tilde{E}_1$ are also symmetrical about the baseline T1 MF $E_{-1}$ and $E_1$, respectively, as shown in Fig. 1(b).
3) $\dot{E}_{-1}$ and $\dot{E}_1$ intersect at 0, and $E_{-1}$ and $E_1$ also intersect at 0.

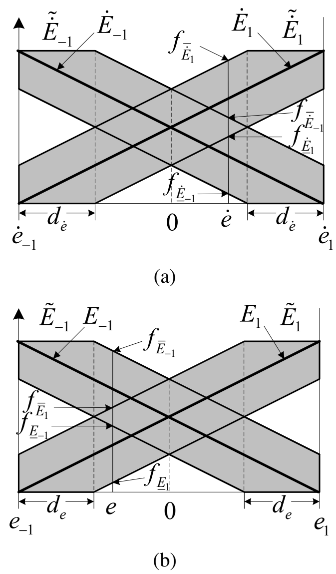



Fig. 1. The fuzzy partitions around the origin. (a) $\dot{e}$ domain, and, (b) $e$ domain.

## III. Equivalent Proportional and Integral Gains of an IT2 Fuzzy PI Controller

It has been established that the T1 fuzzy partitions in Fig. 1 realize the linear PI controller in (1). This section will study how the controller is changed when symmetrical FOUs are introduced to the baseline T1 FLC. To simplify the computation, we only consider the region around the origin bounded by the following inequalities:

$$\dot{e}_{-1} + d_{\dot{e}} \ \le \dot{e} \ \le \ \dot{e}_1 - d_{\dot{e}} \tag{3}$$

$$e_{-1} + d_e \ \le e \ \le \ e_1 - d_e \tag{4}$$

which are motivated by the observations that the robustness improvement occurs mainly when the system output is near the setpoint. A graphical illustration of the region is shown in Fig. 2.

When an input $(\dot{e}, e)$ falls into the shaded region in Fig. 2, only four MFs, $\tilde{\dot{E}}_{-1}$, $\tilde{\dot{E}}_1$, $\tilde{E}_{-1}$ and $\tilde{E}_1$, are fired, and the firing levels are (see Fig. 1):

$$F_{\tilde{\dot{E}}_{-1}} = [f_{\underline{\dot{E}}_{-1}}, f_{\overline{\dot{E}}_{-1}}] = \left[\frac{\dot{e}_1 - d_{\dot{e}} - \dot{e}}{2\dot{e}_1}, \frac{\dot{e}_1 + d_{\dot{e}} - \dot{e}}{2\dot{e}_1}\right] \tag{5}$$

$$F_{\tilde{\dot{E}}_1} = [f_{\underline{\dot{E}}_1}, f_{\overline{\dot{E}}_1}] = \left[\frac{\dot{e} + \dot{e}_1 - d_{\dot{e}}}{2\dot{e}_1}, \frac{\dot{e} + \dot{e}_1 + d_{\dot{e}}}{2\dot{e}_1}\right] \tag{6}$$

$$F_{\tilde{E}_{-1}} = [f_{\underline{E}_{-1}}, f_{\overline{E}_{-1}}] = \left[\frac{e_1 - d_e - e}{2e_1}, \frac{e_1 + d_e - e}{2e_1}\right] \tag{7}$$

$$F_{\tilde{E}_1} = [f_{\underline{E}_1}, f_{\overline{E}_1}] = \left[\frac{e + e_1 - d_e}{2e_1}, \frac{e + e_1 + d_e}{2e_1}\right] \tag{8}$$

Consequently, only four rules, $\tilde{R}^{-1,-1}$, $\tilde{R}^{-1,1}$, $\tilde{R}^{1,-1}$ and $\tilde{R}^{1,1}$, can be fired. Their consequents are:

$$\dot{u}_{-1,-1} = -\dot{e}_1 K_P - e_1 K_I \tag{9}$$

$$\dot{u}_{-1,1} = -\dot{e}_1 K_P + e_1 K_I \tag{10}$$

$$\dot{u}_{1,-1} = \ \ \dot{e}_1 K_P - e_1 K_I \tag{11}$$

$$\dot{u}_{1,1} = \ \ \dot{e}_1 K_P + e_1 K_I \tag{12}$$

Note that we have used the fact that $\dot{e}_{-1} = -\dot{e}_1$ and $e_{-1} = -e_1$.

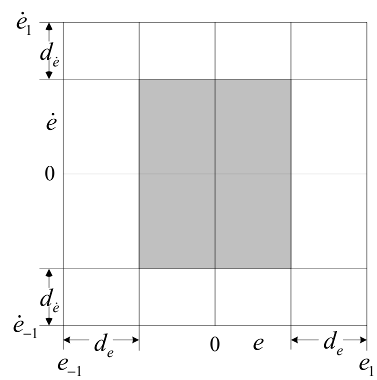


Fig. 2. The region of the input domain determined by (3) and (4).

The firing intervals of the four rules are:

$$\begin{aligned} F_{-1,-1} &= F_{\tilde{\dot{E}}_{-1}} \star F_{\tilde{E}_{-1}} = [\underline{f}_{-1,-1}, \overline{f}_{-1,-1}] \\ &= \left[\frac{(\dot{e}_1 - d_{\dot{e}} - \dot{e})(e_1 - d_e - e)}{4\dot{e}_1 e_1}, \right. \\ &\qquad \left. \frac{(\dot{e}_1 + d_{\dot{e}} - \dot{e})(e_1 + d_e - e)}{4\dot{e}_1 e_1}\right] \end{aligned} \tag{13}$$

$$\begin{aligned} F_{-1,1} &= F_{\tilde{\dot{E}}_{-1}} \star F_{\tilde{E}_1} = [\underline{f}_{-1,1}, \overline{f}_{-1,1}] \\ &= \left[\frac{(\dot{e}_1 - d_{\dot{e}} - \dot{e})(e + e_1 - d_e)}{4\dot{e}_1 e_1}, \right. \\ &\qquad \left. \frac{(\dot{e}_1 + d_{\dot{e}} - \dot{e})(e + e_1 + d_e)}{4\dot{e}_1 e_1}\right] \end{aligned} \tag{14}$$

$$\begin{aligned} F_{1,-1} &= F_{\tilde{\dot{E}}_1} \star F_{\tilde{E}_{-1}} = [\underline{f}_{1,-1}, \overline{f}_{1,-1}] \\ &= \left[\frac{(\dot{e} + \dot{e}_1 - d_{\dot{e}})(e_1 - d_e - e)}{4\dot{e}_1 e_1}, \right. \\ &\qquad \left. \frac{(\dot{e} + \dot{e}_1 + d_{\dot{e}})(e_1 + d_e - e)}{4\dot{e}_1 e_1}\right] \end{aligned} \tag{15}$$

$$F_{1,1} = F_{\tilde{\dot{E}}_1} \star F_{\tilde{E}_1} = [\underline{f}_{1,1}, \overline{f}_{1,1}]$$

$$= \left[ \frac{(\dot{e}+\dot{e}_1-d_{\dot{e}})(e+e_1-d_e)}{4\dot{e}_1 e_1}, \frac{(\dot{e}+\dot{e}_1+d_{\dot{e}})(e+e_1+d_e)}{4\dot{e}_1 e_1} \right] \quad (16)$$

The type-reduced output set of the IT2 fuzzy PI controller, obtained via center-of-sets type reduction [6], is computed as

$$\tilde{\dot{u}} = \frac{F_{-1,-1}\dot{u}_{-1,-1} + F_{-1,1}\dot{u}_{-1,1} + F_{1,-1}\dot{u}_{1,-1} + F_{1,1}\dot{u}_{1,1}}{F_{-1,-1}+F_{-1,1}+F_{1,-1}+F_{1,1}} = [\dot{u}_l, \dot{u}_r] \quad (17)$$

Note that (17) is only an expressive way to describe the type-reduction operation. $\dot{u}_l$ and $\dot{u}_r$, which are the minimum and maximum of $\tilde{\dot{u}}$, are actually computed by the Karnik-Mendel (KM) Algorithms [3], [6] or the Enhanced Karnik-Mendel (EKM) Algorithms [7], [12]. When KM or EKM Algorithms are used, the consequents, $\dot{u}_{ij}$ in (9)-(12), need to be arranged in ascending order. Since there are only four consequents and their ranking depends on $\dot{e}_1 K_P$ and $e_1 K_I$, there are three possible rankings corresponding to the three different relationship between $\dot{e}_1 K_P$ and $e_1 K_I$.

### A. Case 1: $\dot{e}_1 K_P > e_1 K_I$

When $\dot{e}_1 K_P > e_1 K_I$, observe from (9)-(12) that

$$\dot{u}_{-1,-1} < \dot{u}_{-1,1} < 0 < \dot{u}_{1,-1} < \dot{u}_{1,1} \quad (18)$$

To derive closed-form solutions, we further impose the following constraint:

$$\dot{u}_{-1,1} \le \dot{u}_l \le \dot{u}_r \le \dot{u}_{1,-1} \quad (19)$$

As will be shown in Section IV, (19) reduces the shaded region shown in Fig. 2; however, it still ensures that the inputs under consideration are around the origin.

According to KM or EKM Algorithms, (19) indicates that

$$\dot{u}_l = \frac{\overline{f}_{-1,-1}\dot{u}_{-1,-1} + \overline{f}_{-1,1}\dot{u}_{-1,1} + \underline{f}_{1,-1}\dot{u}_{1,-1} + \underline{f}_{1,1}\dot{u}_{1,1}}{\overline{f}_{-1,-1}+\overline{f}_{-1,1}+\underline{f}_{1,-1}+\underline{f}_{1,1}} = \frac{\dot{e}_1(K_P d_{\dot{e}} e_1 - K_P \dot{e} e_1 + K_P \dot{e}_1 d_e + K_I e_1 e)}{e_1\dot{e}_1 + d_{\dot{e}} d_e - \dot{e} d_e} \quad (20)$$

$$\dot{u}_r = \frac{\underline{f}_{-1,-1}\dot{u}_{-1,-1} + \underline{f}_{-1,1}\dot{u}_{-1,1} + \overline{f}_{1,-1}\dot{u}_{1,-1} + \overline{f}_{1,1}\dot{u}_{1,1}}{\underline{f}_{-1,-1}+\underline{f}_{-1,1}+\overline{f}_{1,-1}+\overline{f}_{1,1}} = \frac{\dot{e}_1(K_P d_{\dot{e}} e_1 + K_P \dot{e}_1 d_e + K_P \dot{e} e_1 + K_I e_1 e)}{e_1\dot{e}_1 + d_{\dot{e}} d_e - \dot{e} d_e} \quad (21)$$

Hence, the output of the IT2 FLC is

$$\begin{aligned}\dot{u} &= \frac{\dot{u}_l + \dot{u}_r}{2} \\ &= \frac{\dot{e}_1^2(e_1^2-d_e^2)K_P\dot{e} + \dot{e}_1(\dot{e}_1 e_1^2 + e_1 d_{\dot{e}} d_e)K_I e}{(e_1\dot{e}_1 + d_{\dot{e}} d_e)^2 - d_e^2\dot{e}^2} \\ &= \frac{\dot{e}_1^2(e_1^2-d_e^2)}{(e_1\dot{e}_1 + d_{\dot{e}} d_e)^2 - d_e^2\dot{e}^2}K_P\dot{e} + \frac{\dot{e}_1(\dot{e}_1 e_1^2 + e_1 d_{\dot{e}} d_e)}{(e_1\dot{e}_1 + d_{\dot{e}} d_e)^2 - d_e^2\dot{e}^2}K_I e \\ &\equiv \alpha K_P \dot{e} + \beta K_I e \end{aligned} \quad (22)$$

where

$$\alpha = \frac{\dot{e}_1^2(e_1^2-d_e^2)}{(e_1\dot{e}_1 + d_{\dot{e}} d_e)^2 - d_e^2\dot{e}^2} \quad (23)$$

$$\beta = \frac{e_1\dot{e}_1(\dot{e}_1 e_1 + d_{\dot{e}} d_e)}{(e_1\dot{e}_1 + d_{\dot{e}} d_e)^2 - d_e^2\dot{e}^2} \quad (24)$$

$\alpha K_p$ is the *equivalent Proportional gain* of the resulting IT2 fuzzy PI controller, and $\beta K_I$ is the *equivalent Integral gain*. Observe that:

1) When $\dot{e}_1 K_P > e_1 K_I$, both $\alpha$ and $\beta$ are functions of $\dot{e}$ but not $e$, i.e., the equivalent PI gains change as the input $\dot{e}$ changes.
2) $\alpha$ is always smaller than 1; when $|\dot{e}|$ is small, e.g., $|\dot{e}| \le d_{\dot{e}}$, $\beta$ is also smaller than 1. So, for small inputs (disturbances) around the origin, the equivalent PI gains are smaller than the PI gains of the baseline T1 FLC. Consequently, the same amount of disturbance will cause a smaller control signal change, and hence reduces the risk of oscillation.
3) Because $\frac{\partial \alpha}{\partial d_e} < 0$, $\frac{\partial \beta}{\partial d_e} < 0$, $\frac{\partial \alpha}{\partial d_{\dot{e}}} < 0$, and $\frac{\partial \beta}{\partial d_{\dot{e}}} < 0$ for small $\dot{e}$, generally an increase in $d_e$ and/or $d_{\dot{e}}$ will reduce both $\alpha$ and $\beta$, i.e., larger FOUs will result in smaller equivalent PI gains around the origin, and hence the resulting IT2 FLC is potentially more robust; however, the settling time may increase.
4) Dividing (23) by (24) yields
$$\frac{\alpha}{\beta} = \frac{e_1^2\dot{e}_1 - \dot{e}_1 d_e^2}{e_1^2\dot{e}_1 + e_1 d_{\dot{e}} d_e} < 1 \quad (25)$$
i.e., the equivalent Proportional gain decreases relatively faster compared with the equivalent Integral gain. Observe also that when the FOUs increase, i.e., $d_{\dot{e}}$ and/or $d_e$ increase, $\alpha/\beta$ decreases. Consequently, a larger FOU will increase the damping of the PI controller, and hence reduces overshoots and oscillations. However, this may also increase the settling time.

Knowing $\dot{u}_l$ and $\dot{u}_r$ in (20) and (21), constraint (19) can be re-expressed as

$$|(K_P e_1\dot{e}_1 - K_P d_e \dot{e}_1 + K_I e_1 d_e)\dot{e} + K_I e_1 \dot{e}_1 e| \le K_P\dot{e}_1(\dot{e}_1 - d_{\dot{e}})(e_1 - d_e) - K_I e_1(e_1\dot{e}_1 + d_{\dot{e}} d_e) \quad (26)$$

Equation (26), together with (3) and (4), determines the complete input region in which the equivalent PI gains (23) and (24) are applicable.

### B. Case 2: $\dot{e}_1 K_P < e_1 K_I$

When $\dot{e}_1 K_P < e_1 K_I$,

$$\dot{u}_{-1,-1} < \dot{u}_{1,-1} < 0 < \dot{u}_{-1,1} < \dot{u}_{1,1} \quad (27)$$

Similar to the method adopted in the previous subsection, the following constraint is imposed to deduce closed-form solutions and also ensure that the input region under consideration is around the origin:

$$\dot{u}_{1,-1} \le \dot{u}_l \le \dot{u}_r \le \dot{u}_{-1,1} \quad (28)$$

By repeating the mathematical manipulations described in the previous sub-section, the output of the IT2 fuzzy PI controller is found to be

$$\dot{u} = \alpha' K_P \dot{e} + \beta' K_I e \tag{29}$$

where

$$\alpha' = \frac{e_1 \dot{e}_1 (e_1 \dot{e}_1 + d_{\dot{e}} d_e)}{(e_1 \dot{e}_1 + d_{\dot{e}} d_e)^2 - d_{\dot{e}}^2 e^2} \tag{30}$$

$$\beta' = \frac{e_1^2 (\dot{e}_1^2 - d_{\dot{e}}^2)}{(e_1 \dot{e}_1 + d_{\dot{e}} d_e)^2 - d_{\dot{e}}^2 e^2} \tag{31}$$

The *equivalent Proportional gain* is $\alpha' K_p$ and the *equivalent Integral gain* is $\beta' K_I$. Observe that:

1) When $\dot{e}_1 K_P < e_1 K_I$, both $\alpha'$ and $\beta'$ are functions of $e$ but not $\dot{e}$, i.e., the equivalent PI gains change as the input $e$ changes.
2) $\beta'$ is always smaller than 1; when $|e|$ is small, e.g., $|e| \leq d_e$, $\alpha'$ is also smaller than 1. So, for small inputs (disturbances) around the origin, the equivalent PI gains are smaller than the PI gains of the baseline T1 FLC. Consequently, the same amount of disturbance will cause a smaller control signal change, and hence reduces the risk of oscillation; however, the settling time may increase.
3) Because $\frac{\partial \alpha'}{\partial d_e} < 0$ for small $e$, $\frac{\partial \beta'}{\partial d_e} < 0$, $\frac{\partial \alpha'}{\partial d_{\dot{e}}} < 0$, and $\frac{\partial \beta'}{\partial d_{\dot{e}}} < 0$, generally an increase in $d_e$ and/or $d_{\dot{e}}$ will reduce both $\alpha'$ and $\beta'$, i.e., larger FOUs will result in smaller equivalent PI gains around the origin, and hence the resulting IT2 FLC is potentially more robust; however, the settling time may increase.
4) Dividing (30) by (31) yields

$$\frac{\alpha'}{\beta'} = \frac{e_1 \dot{e}_1^2 + \dot{e}_1 d_e d_{\dot{e}}}{e_1 \dot{e}_1^2 - \dot{e}_1 d_{\dot{e}}^2} > 1 \tag{32}$$

i.e., the equivalent Proportional gain decreases relatively slower compared to the equivalent Integral gain. Observe also that when the FOUs increase, i.e., $d_{\dot{e}}$ and/or $d_e$ increase, $\alpha'/\beta'$ increases. This will decrease the damping of the PI controller, and hence increases the response speed. However, because the magnitudes of both the PI gains are reduced, the output control signal is small in magnitude, and hence the acceleration in response speed may not be obvious.

Similarly, the constraint (28) can be re-expressed as

$$\begin{aligned} &|K_P e_1 \dot{e}_1 \dot{e} + (K_I e_1 \dot{e}_1 + K_P \dot{e}_1 d_{\dot{e}} - K_I e_1 d_{\dot{e}}) e| \leq \\ &\quad K_I e_1 (e_1 - d_e)(\dot{e}_1 - d_{\dot{e}}) - K_P \dot{e}_1 (e_1 \dot{e}_1 + d_{\dot{e}} d_e) \end{aligned} \tag{33}$$

Equation (33), together with (3) and (4), determines the complete input region that the equivalent PI gains (30) and (31) are applicable.

### C. Case 3: $\dot{e}_1 K_P = e_1 K_I$

When $\dot{e}_1 K_P = e_1 K_I$, $\dot{u}_{-1,1} = \dot{u}_{1,-1} = 0$. Similar analysis in the previous two sub-sections cannot be performed here. However, since $\dot{e}_1 K_P = e_1 K_I$ rarely happens in practice, it does not affect the effectiveness of the results in this paper.

## IV. Example

Recent experimental results on IT2 fuzzy PI controller [1], [13], [15] indicate that IT2 fuzzy PI controllers are more robust and are better able to eliminate oscillations. This section utilizes the expressions for the equivalent PI gains as a tool to explain why this is possible.

### A. Equivalent PI Gains

The following simple first-order plus dead-time plant is employed as the nominal system and used to design an IT2 fuzzy PI controller:

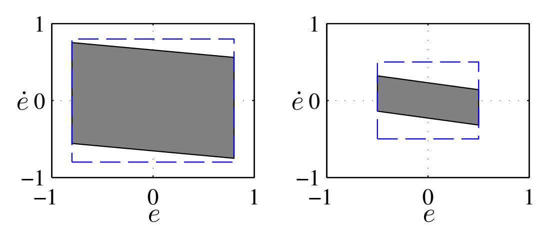
$$G(s) = \frac{K}{\tau s + 1} e^{-Ls} = \frac{1}{10s + 1} e^{-2.5s} \tag{34}$$

Two MFs are used to characterize each input domain, $e$ and $\dot{e}$. The IT2 FSs used here are shown in Fig. 1, where $\dot{e}_1 = e_1 = 1$. Furthermore, $d_{\dot{e}} = d_e \equiv d$ is employed. According to the integral of time absolute error (ITAE) setpoint tracking tuning rule [9], the baseline PI parameters for $G(s)$ are

$$\begin{aligned} \dot{u} &= 0.586K \left(\frac{L}{\tau}\right)^{-0.916} \left[\dot{e} + \frac{1.03 - 0.165\frac{L}{\tau}}{\tau} e\right] \\ &= 2.086\dot{e} + 0.2063e \end{aligned} \tag{35}$$

Hence, consequents of the rules for the IT2 fuzzy PI controller are generated by substituting $K_P = 2.086$ and $K_I = 0.2063$ into (2).

As $\dot{e}_1 K_P > e_1 K_I$, the equivalent PI gains are determined by (22). The closed-form solutions of the equivalent PI gains are derived using the assumption shown in (19). Using (3), (4) and (26), the input regions in which the equivalent PI gains are valid when $d = \{0.2, 0.5\}$ are plotted in Figs. 3(a) and 3(b), respectively. The diagrams indicate that the constraint (26) further restricts the region where the equivalent PI gains are applicable.




(a) (b)

Fig. 3. The input regions where (22) is applicable when (a) $d_{\dot{e}} = d_e = 0.2$; and (b) $d_{\dot{e}} = d_e = 0.5$. The dashed squares are the input regions when the constraint (26) is not considered.

Fig. 4 shows how $\alpha$ and $\beta$ vary with $\dot{e}$ in the range where the equivalence is valid. Observe that the extra degrees of freedom provided by the FOUs result in varying equivalent PI gains. Unlike the baseline T1 FLC whose input-output relationship is linear, the IT2 fuzzy PI controller realizes a non-linear PI control law around the origin. Because both $\alpha$ and $\beta$ are smaller than unity, the equivalent PI gains are smaller than

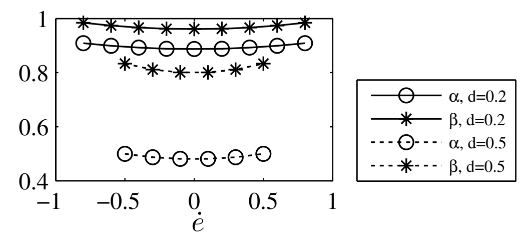


Fig. 4. Relationship between $\alpha$, $\beta$ and $\dot{e}$, where $d_{\dot{e}} = d_e = d$.

the PI gains for the baseline T1 FLC. The deviation from the T1 FLC becomes larger as $d_{\dot{e}}$ increase.

The control surfaces of the baseline T1 FLC and the IT2 FLC with $d = 0.5$ are shown in Figs. 5(a) and 5(b), respectively. Denote the output of the T1 FLC as $\dot{u}_0$, the output of the IT2 FLC with $d = 0.2$ as $\dot{u}_{0.2}$, and the output of the IT2 FLC with $d = 0.5$ as $\dot{u}_{0.5}$. Then, $|\dot{u}_0| - |\dot{u}_{0.2}|$ is shown in Fig. 5(c), and $|\dot{u}_{0.2}| - |\dot{u}_{0.5}|$ is shown in Fig. 5(d). Observe that as the FOU increases, the output becomes smaller in magnitude.

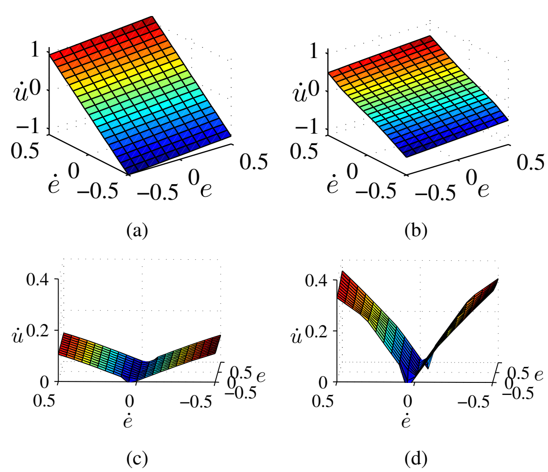


Fig. 5. (a) Control surfaces of the T1 FLC; (b) control surface of the IT2 FLC with $d = 0.5$; (c) $|\dot{u}_0| - |\dot{u}_{0.2}|$; and, (d) $|\dot{u}_{0.2}| - |\dot{u}_{0.5}|$.

## B. Control Performance

In order to examine how the equivalent PI gains correlate with control performances, step responses were obtained using IT2 fuzzy PI controllers for the nominal plant in (34) when $d = 0$ (corresponding to a T1 fuzzy PI controller), $d = 0.2$ and $d = 0.5$. Since the aim is to demonstrate how a change in FOU size would influence[2] the performance of an IT2 fuzzy PI controller, the IT2 FLC is essentially the same as the baseline T1 FLC except for the FOU, and no optimization is employed.

Step responses on the nominal plant are shown in Fig. 6. The corresponding trajectories of $\alpha$ and $\beta$ are shown in Fig. 7. Observe that:

[2]Note that introducing FOUs to a T1 FLC does not necessarily improve its performance. The experiments in this section are only used to illustrate the effect of FOUs.

1) The responses of the two IT2 FLCs are slower than the baseline T1 FLC.
2) $\alpha$ and $\beta$ vary with time because $\dot{e}$ is a function of time.
3) Both $\alpha$ and $\beta$ are smaller than 1. As the FOU increases, both $\alpha$ and $\beta$ decrease.

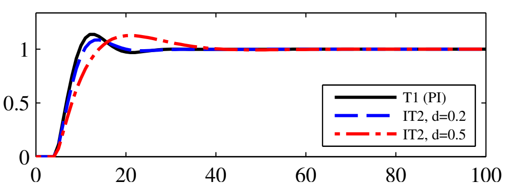



Fig. 6. Step responses of the IT2 and T1 fuzzy PI controllers on the nominal plant, $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{10s+1}e^{-2.5s}$.

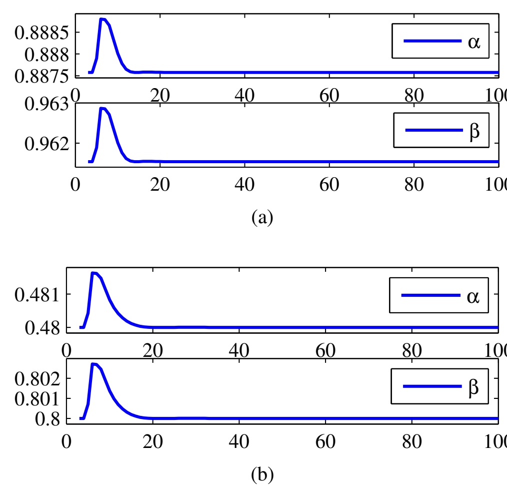


Fig. 7. Trajectories of $\alpha$ and $\beta$ when (a) $d = 0.2$ and (b) $d = 0.5$.

Fig. 8 shows step responses that illustrate how IT2 fuzzy PI controllers ($d = \{0, 0.2, 0.5\}$) cope with parameter uncertainty. Robustness of the IT2 fuzzy PI controllers is tested by varying the static gain and time constant of the process. The ITAEs for the various tests are listed in Table I. Observe from Fig. 8 and Table I that:

1) The larger the FOU, the better the ability of an IT2 FLC to eliminate oscillations about the setpoint [see Figs. 8(a) and 8(c)]. This coincides with the observations in [5], [13], [14], [16]. As explained in Section III-A, this is because a larger FOU increases the damping of the PI controller, and hence reduces overshoots and oscillations.
2) The IT2 fuzzy PI controller may be slower than its T1 counterpart when the dynamics of the plant is slow [Figs. 8(b) and 8(d)]; particularly, the settling time is longer. This is because the smaller equivalent PI gains result in smaller control signal, whereas we need a large control signal when the dynamics of a plant is slow. In practice, if we know the system dynamics is slow, or

TABLE I
ITAEs OF THE THREE FLCs IN THE FIVE STEP RESPONSES.

| | $k=1,\tau=10$ | $k=2,\tau=10$ | $k=0.5,\tau=10$ | $k=1,\tau=5$ | $k=1,\tau=20$ |
|---|---|---|---|---|---|
| T1 FLC (Linear PI) | 38 | 308 | 85 | 104 | 129 |
| IT2 FLC, $d=0.2$ | 36 | 160 | 93 | 73 | 152 |
| IT2 FLC, $d=0.5$ | 85 | 70 | 157 | 33 | 336 |

the IT2 fuzzy PI controller is tuned by an optimization algorithm, very likely the PI gains around the origin will be increased to achieve a better compromise between robustness and response speed.

All observations in this section coincide with the theoretical results presented in Section III-A.

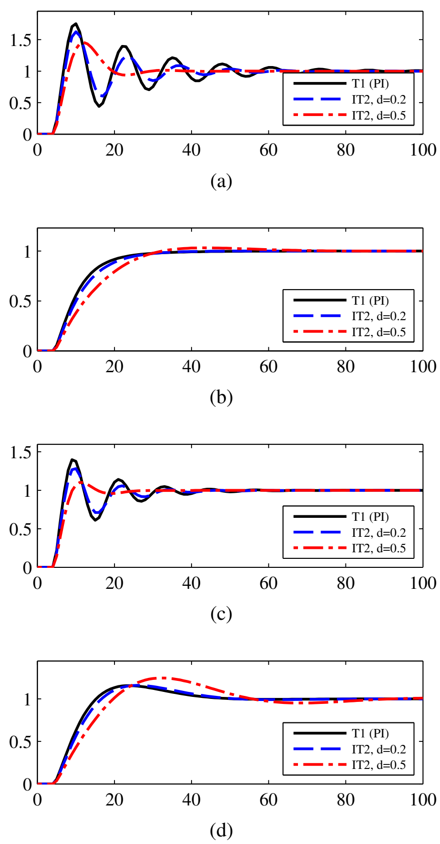



Fig. 8. Step responses of the IT2 and T1 fuzzy PI controllers for different plant parameters. (a) $k=2,\tau=10$; (b) $k=0.5,\tau=10$; (c) $k=1,\tau=5$; and, (d) $k=1,\tau=20$.

## V. Conclusions

The concept of equivalent PI gains of an IT2 fuzzy PI controller has been introduced in this paper. It is shown that an IT2 fuzzy PI controller may implement a variable gain PI control law around the origin. Generally, the equivalent PI gains are smaller than those of the baseline T1 FLC, and they decrease as the FOU increases. Because smaller PI gains generate smaller control signals, this explains why IT2 FLCs may be more robust to disturbances and why their response may be slower. The findings provide theoretical explanation for the experimental observations suggesting that an IT2 fuzzy PI controller is better able to cope with uncertainty and eliminate steady-state oscillations. Additionally, because closed-form solutions of the equivalent PI gains have been obtained,

1) Given a good IT2 FLC, we can examine its PI gains around the origin and use them to guide the design of robust PI controllers; and,
2) Given a good PI controller, we can tune the parameters of an IT2 FLC so that its behavior approximates the PI controller.

In other words, the results in this paper enable us to connect traditional PI controllers with IT2 fuzzy PI controllers for the first time. Our future research is to explore how this connection can be used for better PI controller design.

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