Examining the Continuity of Type-1 and Interval Type-2 Fuzzy Logic Systems

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Abstract— **This paper studies the continuity of the input-output mappings of fuzzy logic systems (FLSs), including both type-1 (T1) and interval type-2 (IT2) FLSs. We show that a T1 FLS being an universal approximator is equivalent to saying that a T1 FLS has a continuous input-output mapping. We also derive the condition under which a T1 FLS is discontinuous. For IT2 FLSs using Karnik-Mendel type-reduction and center-of-sets defuzzification, we derive the conditions under which continuous and discontinuous input-output mappings can be obtained. Our results will be very useful in selecting the parameters of the membership functions to achieve a desired continuity (e.g., for most traditional modeling and control applications) or discontinuity (e.g., for hybrid and switched systems modeling and control).**

I. Introduction

Modeling and control is the most widely used application of both type-1 fuzzy logic systems (FLSs) [3], [24], [33], [42], [43] and interval type-2 (IT2) FLSs [11], [26], [31], [39]–[41]. Essentially, a FLS implements a function representing a mapping between inputs and outputs.

In many cases, continuous and smooth input-output mapping is desired for a FLS, because most physical systems are continuous, and a continuous and smooth control surface is usually more favorable in terms of stability and performance, e.g., Wu and Tan [37], [39], [40] and Jammeh et al. [12] have shown that an IT2 fuzzy logic controller may outperform its T1 counterpart because it gives a smoother control surface, especially in the region around the steady state (both the error and the change of error approach 0). So, for such applications, we need to avoid abrupt changes, especially discontinuities, in the input-output mappings. It would be very beneficial to find the conditions under which a FLS gives a continuous input-output mapping so that we can ensure a continuous mapping when it is desired.

Nevertheless, discontinuous FLSs may be very useful in hybrid and switched systems [22], [30] modeling and control, which is becoming increasingly popular recently due to the wide applications of computers and digital controllers. Hybrid systems [4] are finite-state machines coupled with controllers and plants modeled by differential or difference equations. They arise whenever logical decision making is mixed with the generation of continuous-valued control laws. Switched systems are an important class of hybrid systems. They consist of "*a finite number of continuous-time subsystems and a logical rule that orchestrates switching between them*" (p. 3, [30]). FLSs have been extensively used in hybrid and switched systems modeling and control [1], [10], [21], [23], [28], [29]; however, to the authors' best knowledge, all these approaches consider each continuous-time subsystem separately. For example, considered a simple room-temperature control problem: If the temperature is lower than 10°C, then the heater is on; otherwise, the heater is off. Clearly, the differential equations representing the room-temperature dynamics are different in the two states (with or without heater). Traditional modeling approach for switched systems would model the two discrete states separately; however, a discontinuous FLS may be designed to have a discontinuity at 10°C to model the two discrete states by a single FLS. This would simplify the model understanding and representation. This paper will shed some light on how to do this.

Surprisingly, though fuzzy sets have been used for more than 40 years, little research has been conducted directly on the continuity of FLSs. Many results have shown that T1 FLSs are universal approximators [5]–[7], [17]–[19], [32]. This is equivalent to saying that a T1 FLS can implement any real continuous function, as we will prove in this paper; however, it is still unclear whether and when a T1 FLS can implement a discontinuous real function. Furthermore, to the best of the authors' knowledge, no researcher has considered the continuity of IT2 FLSs.

The rest of this paper is organized as follows: Section II studies the continuity of T1 FLSs. Section III studies the continuity of IT2 FLSs with Karnik-Mendel type-reduction and center-of-sets defuzzification, the most popular IT2 FLSs in practice. Finally, Section IV draws conclusions.

II. Continuity of T1 FLSs

This section studies the continuity of T1 FLSs. First, properties of continuous functions are reviewed.

A. Properties of Continuous Functions

Definition 1: A single-variable function $f(x)$ is *continuous* at c if and only if $f(x)$ is defined at c, and whenever x is infinitely close to c, $f(x)$ is infinitely close to $f(c)$, i.e., for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$. □

Recall the following facts about continuous functions from elementary calculus [8], [14]:

1) If $f(x)$ is differentiable at c, then it is continuous at c.
2) Suppose both $f_1(x)$ and $f_2(x)$ are continuous at c:
   a) For any constant k, the function $k \cdot f_1(x)$ is continuous at c.
   b) $f_1(x) + f_2(x)$ is continuous at c.
   c) $f_1(f_2(x))$ is continuous at c if $f_1(x)$ is continuous at $f_2(c)$.

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d) $f_1(x)/f_2(x)$ is continuous at c if $f_2(c) \neq 0$.

We distinguish between two types of discontinuities in this paper.

Definition 2: A function $f(x)$ has a *gap discontinuity* at c if $f(c)$ is *undefined*. □

For example, $f_1(x)/f_2(x)$ has a gap at c if $f_2(c) = 0$.

Definition 3: A function $f(x)$ has a *jump discontinuity* at c if $f(c)$ is *defined* but it does not satisfy the continuity defined by Definition 1, i.e., both $f(c)$ and $f(c+\delta)$ are defined, but $f(c) \neq f(c+\delta)$, where δ is an arbitrarily small positive or negative number. □

For example, $f(x) = \begin{cases} 2, & x<0 \\ 3, & x\geqslant 0 \end{cases}$ has a jump discontinuity at $x = 0$.

A multi-variable continuous function $f(\mathbf{x})$ of an M-dimension input $\mathbf{x} = (x_1, x_2, ..., x_M)$ is defined as:

Definition 4: A multi-variable function $f(\mathbf{x})$ is continuous at $\mathbf{c} = (c_1, c_2, ..., c_M)$ if and only if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\max\limits_{m=1,...,M} |x_m - c_m| < \delta \Rightarrow |f(\mathbf{x}) - f(\mathbf{c})| < \epsilon$. □

The facts about continuous single-variable functions, introduced earlier in this subsection, also hold for continuous multi-variable functions.

B. Structure of the T1 FLSs

For simplicity we consider only multi-antecedent single-consequent T1 FLSs in this paper; however, our results can be easily extended to multi-antecedent multi-consequent T1 FLSs, because the latter can be decomposed into several multi-antecedent single-consequent T1 FLSs [20].

The T1 FLS has M inputs, $\{x_m\}_{m=1,2,...,M}$, and one output, y. Assume the m^{th} input has N_m membership functions (MFs) in its universe of discourse, $\mathbb{X}_m$. Denote the n^{th} MF in the m^{th} input domain as X_{mn}. A complete rulebase with all possible combinations of the input MFs consists of $K = \prod_{m=1}^{M} N_m$ rules in the form of:

R^k: IF x_1 is $X_{1,n_{1k}}$ and ... and x_M is $X_{M,n_{Mk}}$, THEN y is y_k, $\quad n_{ik} = 1, 2, ..., N_i,\ k = 1, 2, ..., K$

where y_k is a constant, and generally it is different for different rules. An example rulebase for a T1 FLS with two inputs ($M = 2$) and three MFs for each input ($N_1 = N_2 = 3$) is shown in Table I. Note that this T1 FLS can be viewed as the simplest TSK model, where each rule consequent is represented by a crisp number. It can also be viewed as a Mamdani model with centroid (or height) defuzzification [26], i.e., y_k represents the centroid (or point with the maximum membership degree) of the consequent T1 FS of the k^{th} rule. Though the rulebase looks simple, it actually represents the most frequently used T1 FLS in practice.

TABLE I

AN EXAMPLE RULEBASE OF A T1 FLS WITH TWO INPUTS AND THREE MFS FOR EACH INPUT.

$x_1 \setminus x_2$	X_{21}	X_{22}	X_{23}
X_{11}	y_1	y_2	y_3
X_{12}	y_4	y_5	y_6
X_{13}	y_7	y_8	y_9

For an input $\mathbf{x} = (x_1, x_2, ..., x_M)$, the output of a T1 FLS with the above structure is computed as:

$$y(\mathbf{x}) = \frac{\sum_{k=1}^{K} f_k y_k}{\sum_{k=1}^{K} f_k} \tag{1}$$

where f_k is the firing level of $\mathbf{x}$ for the k^{th} rule, computed by a t-norm, i.e.,

$$f_k = \mu_{X_{1,n_{1k}}}(x_1) \star \mu_{X_{2,n_{2k}}}(x_2) \star \cdots \star \mu_{X_{M,n_{Mk}}}(x_M) \tag{2}$$

Only minimum and product t-norms [15] are considered in this paper since they are the most frequently used ones in practice.

In this paper we consider only continuous fuzzy sets (FSs) as MFs because discontinuous T1 FSs are almost never used in modeling and control.

Definition 5: A T1 FS X is continuous if and only if its MF, $\mu_X(x)$, is a continuous function of x. □

C. Universal Approximators

Many authors have shown that various configurations of T1 FLSs are universal approximators [5]–[7], [17]–[19], [32], i.e., a T1 FLS can uniformly approximate any real continuous function on a compact domain to any degree of accuracy. For example, Wang and Mendel [32] proved that T1 FLSs with Gaussian MFs, product t-norm and centroid defuzzification are universal approximators; Kreinovich et al. [19] further showed that such T1 FLSs are universal approximators for a smooth function and also its derivatives, i.e., not only the smooth function is approximated by the T1 FLS, but also its derivatives; Castro [7] showed that T1 FLSs with Gaussian, triangular or trapezoidal MFs, any t-norm and any practical defuzzification method are universal approximators; and, Kosko [17] showed that all additive T1 FLSs[1] [16] are universal approximators. Kreinovich et al. [18] also gave a comprehensive review of many such results.

Intuitively, a T1 FLS must realize a continuous input-output mapping in order to approximate a continuous function to any degree of accuracy. This conjecture is mathematically proved in the following:

Theorem 1: A universal approximator $f(\mathbf{x})$ of a continuous function $g(\mathbf{x})$ must be continuous. □

The proof of Theorem 1, as well as proofs for all other theorems in this paper, are given in the Appendix.

So far, we have shown that as long as a T1 FLS is an universal approximator, it is continuous. According to Castro [7], T1 FLSs with Gaussian, triangular or trapezoidal MFs, any t-norms and centroid defuzzification are universal approximators, and hence they are continuous. However, there are still two questions remaining unanswered:

1) Are T1 FLSs with arbitrary continuous MFs (not necessarily Gaussian, triangular or trapezoidal) continuous?
2) In order to be an universal approximator, the T1 FSs must cover all input domains completely. What if there are gaps in at least one input domain?

These two questions are considered next.

[1] Additive T1 FLSs [16] use summation (instead of maximum, as suggested by the Extension Principle [15]) to combine the scaled consequent FSs and then use centroid type-reduction to obtain a crisp output.

D. Continuity of T1 FLSs

Consider the T1 FLS structure introduced in Section II-B.

Theorem 2: The T1 FLS $y(\mathbf{x})$ is continuous at $\mathbf{c} = (c_1, c_2, ..., c_M)$ if and only if $\max_{n=1,2,...,N_m} \mu_{X_{mn}}(c_m) > 0$ for $\forall m = 1, 2, ...M$, i.e., every c_m is covered by some continuous T1 FSs. $\square$

Note that in this paper "c_m is covered by some continuous T1 FSs" means that the membership grade of c_m on at least one of the T1 FSs is larger than 0, e.g., in the middle column of Fig. 1, $x_1 = 0$ is covered by X_{12}, but $x_1 = \pm 0.5$ are not covered.

Theorem 3: The T1 FLS $y(\mathbf{x})$ has a gap discontinuity at $\mathbf{c} = (c_1, c_2, ..., c_M)$ if and only if there exists a c_m such that $\max_{n=1,2,...,N_m} \mu_{X_{mn}}(c_m) = 0$, i.e., there is at least one c_m not covered by any continuous T1 FS in its domain. $\square$

Observe that T1 FLSs cannot have jump discontinuities, because Theorems 2 and 3 have covered all possible T1 FLSs.

For practical T1 FLSs, usually all inputs domains are fully covered by continuous T1 FSs, and hence the T1 FLSs are continuous. That's why people have not paid much attention to the continuity of T1 FLSs; however, the case is quite different and complicated for IT2 FLSs, as we will see in Section III.

E. Examples

Examples demonstrating Theorems 2 and 3 are presented in this subsection.

Fig. 1 shows three input-output mappings of T1 FLSs with only one input. The rulebase is shown in Table II, and product t-norm is used. The numbers in Table II are chosen only for illustration purpose. The first row of Fig. 1 shows the three MFs in the input domain and the second row the corresponding input-output mappings. Observe that:

1) When the input MFs fully cover the input domain, as shown in the first column of Fig. 1, the corresponding input-output mapping is continuous, as indicated by Theorem 2.
2) When at least one point in the input domain is not covered by the MFs, the corresponding input-output mapping has gap discontinuities, as shown in the second and third columns of Fig. 1. These results are consistent with Theorem 3.
3) The gaps in the output domain are determined by the uncovered intervals in the input domain, e.g., as shown in the second column of Fig. 1, x_1 has gaps at $x_1 = \pm 0.5$, and hence its input-output mapping also has gap discontinuities at $x_1 = \pm 0.5$, as indicated by Theorem 3. Similarly, as shown in the third column of Fig. 1, x_1 has gaps at $x_1 = [-0.6, -0.3] \cup [0.3, 0.6]$, and hence its input-output mapping also has gap discontinuities at $x_1 = [-0.6, -0.3] \cup [0.3, 0.6]$.

TABLE II
THE RULEBASE FOR THE T1 FLSS SHOWN IN FIG. 1.

x_1	X_{11}	X_{12}	X_{13}
y	1	2	3

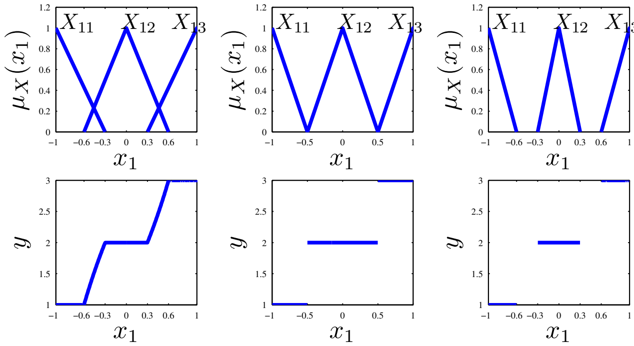

Fig. 1. Example input-output mappings of T1 FLSs with only one input.

Fig. 2 shows the input-output mappings of three T1 FLSs with two inputs. The rulebase is shown in Table III, and product t-norm is used. Again, the numbers in Table III are chosen only for illustration purpose. Observe from Fig. 2 that:

1) When the input MFs fully cover the input domains, as shown in the first column of Fig. 2, the corresponding input-output mapping is continuous, as indicated by Theorem 2.
2) When at least one point in the input domain is not covered by the MFs, the corresponding input-output mapping has gap discontinuities, as shown in the last two columns of Fig. 2. These results are consistent with Theorem 3.
3) The gaps in the output domain are determined by the uncovered intervals in the input domains, e.g., in the second column of Fig. 2 there are two uncovered points $x_2 = \pm 0.5$, and hence the input-output mapping has gap discontinuities at $x_2 = \pm 0.5$, as indicated by Theorem 3. Similarly, as shown in the third column of Fig. 2, both x_1 and x_2 are uncovered at $[-0.6, -0.3]$ and $[0.3, 0.6]$, and hence the input-output mapping has gap discontinuities at $[-0.6, -0.3]$ and $[0.3, 0.6]$ in both x_1 and x_2 domains.

TABLE III
THE RULEBASE FOR THE T1 FLSS SHOWN IN FIG. 2.

$x_1 \setminus x_2$	X_{21}	X_{22}	X_{23}
X_{11}	1	2	3
X_{12}	4	5	6
X_{13}	7	8	9

III. CONTINUITY OF IT2 FLSS

Karnik-Mendel (KM) type-reduction and center-of-sets defuzzification [13], [26], [36] are so far the most popular type-reduction and defuzzification method for IT2 FLSs. The continuity of such IT2 FLSs is studied in this section.

A. Structure of the IT2 FLS

Again we consider only multi-antecedent single-consequent IT2 FLSs in this section; however, our results can be easily extended to multi-antecedent multi-consequent IT2 FLSs, because the latter can be decomposed into several multi-antecedent single-consequent IT2 FLSs.

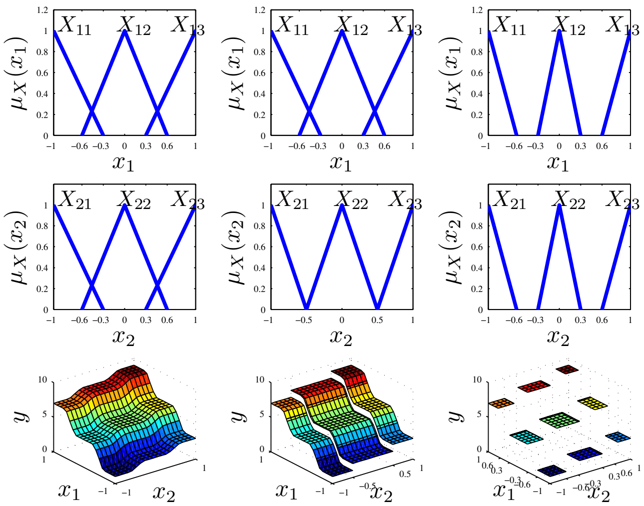

Fig. 2. Example input-output mappings of T1 FLSs with two inputs.

An example IT2 FS $\tilde{X}_{mn}$ is shown in Fig. 3. Its upper membership function (UMF) is denoted $\overline{X}_{mn}$ and lower membership function (LMF) $\underline{X}_{mn}$.

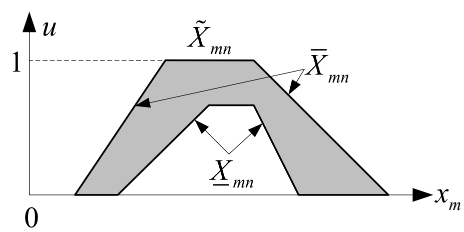

Fig. 3. An IT2 FS $\tilde{X}_{mn}$ and its upper membership function (UMF) $\overline{X}_{mn}$ and lower membership function (LMF) $\underline{X}_{mn}$. Shaded area is the footprint of uncertainty (FOU).

The IT2 FLS has M inputs, $\{x_m\}_{m=1,2,...,M}$, and one output, y. Assume the m^{th} input has N_m MFs in its universe of discourse $\mathbb{X}_m$. Denote the n^{th} MF in the m^{th} input domain as $\tilde{X}_{mn}$. A complete rulebase with all possible combinations of the input MFs consists of $K = \prod_{m=1}^{M} N_m$ rules in the form of:

$\tilde{R}^k$: IF x_1 is $\tilde{X}_{1,n_{1k}}$ and ... and x_M is $\tilde{X}_{M,n_{Mk}}$, THEN y is $[\underline{y}_k, \overline{y}_k]$, $\quad n_{ik} = 1, 2, ..., N_i$, $k = 1, 2, ..., K$

where $[\underline{y}_k, \overline{y}_k]$ is a constant interval, and generally it is different for different rules. Note that this IT2 FLS can be viewed as a Mamdani model with center-of-sets type-reduction and centroid defuzzification [26], i.e., $[\underline{y}_k, \overline{y}_k]$ represents the centroid of the consequent IT2 FS of the k^{th} rule. When $\underline{y}_k = \overline{y}_k$, this rulebase represents the simplest TSK model, where each rule consequent is represented by a crisp number. Again, this rulebase represents the most commonly used IT2 FLSs in practice. An example rulebase for an IT2 FLS with two inputs ($M = 2$) and three MFs for each input ($N_1 = N_2 = 3$) is shown in Table IV.

When KM type-reduction and center-of-sets defuzzification are used, the output of an IT2 FLS with the above structure for an input $\mathbf{x} = (x_1, x_2, ..., x_M)$ is computed as [26]:

$$y(\mathbf{x}) = \frac{y_l(\mathbf{x}) + y_r(\mathbf{x})}{2} \tag{3}$$

where

$$y_l(\mathbf{x}) = \frac{\sum_{k=1}^{k_l} \overline{f}_k \underline{y}_k + \sum_{k=k_l+1}^{K} \underline{f}_k \underline{y}_k}{\sum_{k=1}^{k_l} \overline{f}_k + \sum_{k=k_l+1}^{K} \underline{f}_k} \tag{4}$$

$$y_r(\mathbf{x}) = \frac{\sum_{k=1}^{k_r} \underline{f}_k \overline{y}_k + \sum_{k=k_r+1}^{K} \overline{f}_k \overline{y}_k}{\sum_{k=1}^{k_r} \underline{f}_k + \sum_{k=k_r+1}^{K} \overline{f}_k} \tag{5}$$

in which $[\underline{f}_k, \overline{f}_k]$ is the firing interval of the k^{th} rule, i.e.,

$$\underline{f}_k = \mu_{\underline{X}_{1,n_{1k}}}(x_1) \star \mu_{\underline{X}_{2,n_{2k}}}(x_2) \star \cdots \star \mu_{\underline{X}_{M,n_{Mk}}}(x_M) \tag{6}$$

$$\overline{f}_k = \mu_{\overline{X}_{1,n_{1k}}}(x_1) \star \mu_{\overline{X}_{2,n_{2k}}}(x_2) \star \cdots \star \mu_{\overline{X}_{M,n_{Mk}}}(x_M) \tag{7}$$

Observe that both $\underline{f}_k$ and $\overline{f}_k$ are continuous functions when all IT2 MFs are continuous. Note also that $\{\underline{y}_k\}$ and $\{\overline{y}_k\}$ have been sorted in ascending order in (4) and (5), respectively. The *switch points* k_l and k_r are determined by the KM Algorithms [13], [26] or the Enhanced KM (EKM) Algorithms [34]–[36], and they satisfy:

$$\underline{y}_{k_l} \leqslant y_l(\mathbf{x}) \leqslant \underline{y}_{k_l+1} \tag{8}$$

$$\overline{y}_{k_r} \leqslant y_r(\mathbf{x}) \leqslant \overline{y}_{k_r+1} \tag{9}$$

Only continuous IT2 FSs are of interest in this paper, which are defined as follows:

Definition 6: An IT2 FS $\tilde{X}$ is continuous if and only if both its UMF and its LMF are continuous T1 FSs. □

The continuity of the IT2 FLS is more interesting and complicated than the T1 FLS because, unlike the T1 FLS introduced in Section II-B, the output of the IT2 FLS does not have a closed-form solution. Furthermore, the KM algorithms for type-reduction involve switch points, which give the impression of discontinuity.

TABLE IV
AN EXAMPLE RULEBASE OF AN IT2 FLS WITH TWO INPUTS AND THREE MFS FOR EACH INPUT.

$x_1 \setminus x_2$	$\tilde{X}_{21}$	$\tilde{X}_{22}$	$\tilde{X}_{23}$
$\tilde{X}_{11}$	$[\underline{y}_1, \overline{y}_1]$	$[\underline{y}_4, \overline{y}_4]$	$[\underline{y}_7, \overline{y}_7]$
$\tilde{X}_{12}$	$[\underline{y}_2, \overline{y}_2]$	$[\underline{y}_5, \overline{y}_5]$	$[\underline{y}_8, \overline{y}_8]$
$\tilde{X}_{13}$	$[\underline{y}_3, \overline{y}_3]$	$[\underline{y}_6, \overline{y}_6]$	$[\underline{y}_9, \overline{y}_9]$

B. Continuity of IT2 FLSs

Two theorems on the discontinuities of IT2 FLSs are introduced next.

Theorem 4: The IT2 FLS has a *gap discontinuity* at $\mathbf{c}$ if and only if $\exists c_m$ such that $\max_{n=1,2,...,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$, i.e., there exist at least one c_m not covered by the UMFs. □

Theorem 5: The IT2 FLS has a *jump discontinuity* at $\mathbf{c} = \{c_1, c_2, ..., c_M\}$ if and only if:

1) $\max_{n=1,2,...,N_m} \mu_{\overline{X}_{mn}}(x_m) > 0$ for $\forall x_m$, i.e., the input domain is fully covered by the UMFs; and,

2) $\exists c_m$ such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, i.e., there exists at least one c_m not covered by the LMFs; and,
3) There exists an $m' \neq m$ such that, the minimum $\underline{y}_k$ and/or maximum $\overline{y}_k$ for all fired rules (i.e., those rules with $\overline{f}_k > 0$) changes as $c_{m'}$ changes to $c_{m'}+\delta$, where δ is an arbitrarily small positive or negative number. □

Theorems 4 and 5 suggest that an IT2 FLS can have both gap and jump discontinuities, whereas a T1 FLS can only have gap discontinuities.

The third criterion in Theorem 5 requires $m' \neq m$, i.e., there must be at least two inputs in order to have jump discontinuities. Hence, we have the following:

Lemma 1: An IT2 FLS with only one input does not have jump discontinuities. □

By finding the complement of Theorems 4 and 5, we have the following necessary and sufficient conditions of a continuous IT2 FLS:

Theorem 6: The IT2 FLS is continuous at $\mathbf{c} = \{c_1, c_2, ..., c_M\}$ if and only if:

1) $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall m$; or,
2) For every m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, the minimum $\underline{y}_k$ and maximum $\overline{y}_k$ of all fired rules do not change as any $c_{m'}$ $(m' \neq m)$ changes to $c_{m'}+\delta$, where δ is an arbitrarily small positive or negative number. □

Since the second condition of Theorem 6 is more difficult to test than the first one, we suggest that practitioners who want to avoid both gap and jump discontinuities should focus on satisfying the first condition. Essentially, the first condition of Theorem 6 says that *an IT2 FLS is continuous as long as its input domain is fully covered by* both *the UMFs and the LMFs*. It is not a tight constraint on the shapes of the IT2 FS MFs, e.g., it is satisfied by all IT2 FLSs in [2], [11], [27], [38], [40], [41]; so, it should not limit the modeling power of IT2 FLSs.

C. Examples

Examples demonstrating Theorems 4-6 are presented in this subsection.

Fig. 4 shows three input-output mappings of IT2 FLSs with only one input. The first row shows the three MFs in the input domain and the second row the corresponding input-output mappings. The corresponding rulebase is given in Table V. Observe from Fig. 4 that:

1) When both the input UMFs and the input LMFs fully cover the input domain, as shown in the first column of Fig. 4, the corresponding input-output mapping is continuous, as indicated by Theorem 6.
2) When the input domain is fully covered by the UMFs but at least one point in the input domain is not covered by the LMFs, as shown in the middle column of Fig. 4, the corresponding input-output mapping is still continuous for the 1-input case, because it does not satisfy the third criterion of Theorem 5.
3) When the input UMFs do not fully cover the input domain, as shown in the last column of Fig. 4, the corresponding input-output mapping has gap discontinuities, as indicated by Theorem 4.

Fig. 4 demonstrates that an IT2 FLS with only one input cannot have jump discontinuities, as suggested by Lemma 1; however, that IT2 FLS can still have gap discontinuities.

TABLE V
THE RULEBASE FOR THE IT2 FLSS SHOWN IN FIG. 4.

x_1	$\tilde{X}_{11}$	$\tilde{X}_{12}$	$\tilde{X}_{13}$
y	$[0.8, 1.2]$	$[1.8, 2.2]$	$[2.8, 3.2]$

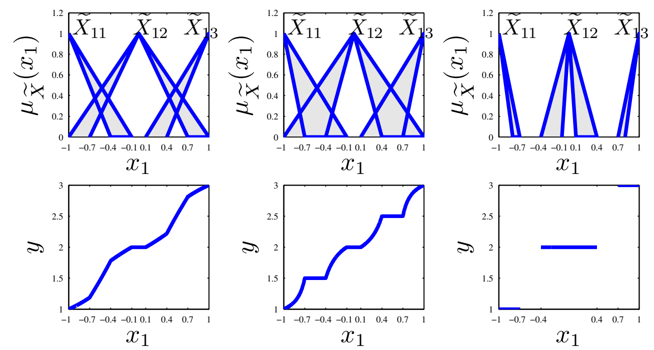

Fig. 4. Example input-output mappings of IT2 FLSs with only one input.

Fig. 5 shows three input-output mappings of IT2 FLSs with two inputs. The first two rows show the input MFs and the third row the corresponding input-output mappings. The rulebase is given in Table VI. Observe from Fig. 5 that:

1) When both the input UMFs and the input LMFs fully cover the input domain, as shown in the first column of Fig. 5, the corresponding input-output mapping is continuous, as indicated by Theorem 6.
2) When the input domain is fully covered by the UMFs but at least one point in the input domain is not covered by the LMFs, as shown in the middle column of Fig. 5, the corresponding input-output mapping has jump discontinuities (e.g., when $x_1 = \pm 0.1$ and $x_2 \in [0.4, 0.7]$), as indicated by Theorem 5. Observe also that through it is the x_2 domain that is not fully covered by the LMFs, the jump discontinuities happen in the domain of x_1.
3) When the input UMFs do not fully cover the input domain, as shown in the last column of Fig. 5, the corresponding input-output mapping has gap discontinuities, as indicated by Theorem 4. Note that the x_2 domain is not covered by the UMFs, so the gap discontinuities happen in the x_2 domain, which is also indicated by Theorem 4.

TABLE VI
THE RULEBASE FOR THE IT2 FLSS SHOWN IN FIG. 5.

$x_1 \setminus x_2$	$\tilde{X}_{21}$	$\tilde{X}_{22}$	$\tilde{X}_{23}$
$\tilde{X}_{11}$	$[0.8, 1.2]$	$[1.8, 2.2]$	$[2.8, 3.2]$
$\tilde{X}_{12}$	$[3.8, 4.2]$	$[4.8, 5.2]$	$[5.8, 6.2]$
$\tilde{X}_{13}$	$[6.8, 7.2]$	$[7.8, 8.2]$	$[8.8, 9.2]$

Theorem 4 is intuitive. Next the second column of Fig. 5 is used as an example to explain in detail how the jump

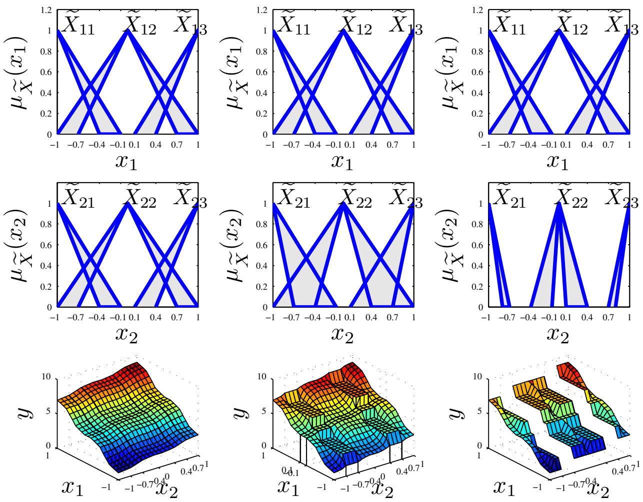

Fig. 5. Example input-output mappings of IT2 FLSs with two inputs.

discontinuities suggested by Theorem 5 are generated. A more detailed plot of the input-output mapping shown in the second column of Fig. 5 is depicted in Fig. 6. Observe that when $x_1 = \pm 0.1$, there are jump discontinuities at $x_2 \in [-0.7, -0.4] \cup [0.4, 0.7]$. The reason is analyzed next.

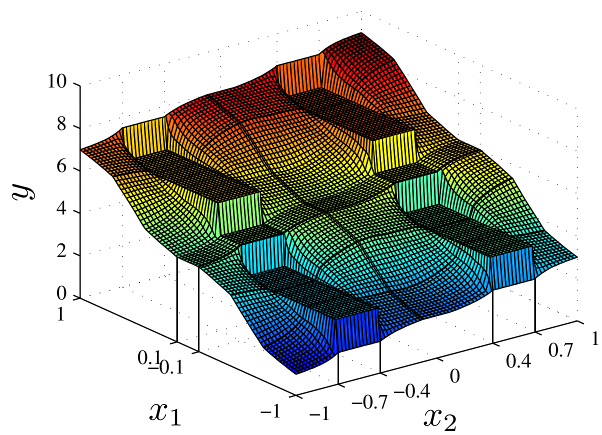

Fig. 6. A detailed illustration of jump continuities. The input MFs are shown in the second column of Fig. 5.

Observe from the second column of Fig. 5 that $\max_{n=1,2,3} \mu_{\overline{X}_{1n}}(x_1) > 0$ and $\max_{n=1,2,3} \mu_{\overline{X}_{2n}}(x_2) > 0$, i.e., the first criterion in Theorem 5 is satisfied. Consider $x_2 = 0.4$, where $\max_{n=1,2,3} \mu_{\underline{X}_{2n}}(x_2) = 0$, i.e., the second criterion of Theorem 5 is also satisfied. Further consider the case that x_1 changes from 0.1 to $0.1 + \delta$, where $\delta > 0$ is an arbitrarily small number:

1) When $x_1 = 0.1$ and $x_2 = 0.4$, the firing intervals of the antecedents and rules are given in Table VII. Observe that only the two rules with antecedents $(\tilde{X}_{12}, \tilde{X}_{22})$ and $(\tilde{X}_{12}, \tilde{X}_{23})$ are fired, and the lower bounds of both firing intervals are 0; hence, from the KM algorithms,

$$y_l(\mathbf{x}) = \frac{.54 \times 4.8 + 0 \times 5.8}{.54 + 0} = 4.8 \tag{10}$$

$$y_r(\mathbf{x}) = \frac{0 \times 5.2 + .3 \times 6.2}{0 + .3} = 6.2 \tag{11}$$

$$y(\mathbf{x}) = \frac{4.8 + 6.2}{2} = 5.5 \tag{12}$$

2) When $x_1 = 0.1 + \delta$ and $x_2 = 0.4$, the firing intervals of the antecedents and rules are given in Table VIII. Observe that only four rules are fired, and the lower bounds of all firing intervals are again 0. Observe also that the minimum $\underline{y}_k$ for all fired rules is 4.8 in both Table VII and VIII; however, the maximum $\overline{y}_k$ for all fired rules changes from 6.2 to 9.2. So, according to Theorem 5, there must be a jump discontinuity. Indeed,

$$\begin{aligned} y_l(\mathbf{x}) &= \frac{(.54 - .6\delta) \times 4.8 + 0 \times 5.8 + 0 \times 7.8 + 0 \times 8.8}{.54 - .6\delta + 0 + 0 + 0} \\ &= 4.8 \end{aligned} \tag{13}$$

$$\begin{aligned} y_r(\mathbf{x}) &= \frac{0 \times 5.2 + 0 \times 6.2 + 0 \times 8.2 + \delta/2.7 \times 9.2}{0 + 0 + 0 + \delta/2.7} \\ &= 9.2 \end{aligned} \tag{14}$$

$$y(\mathbf{x}) = \frac{4.8 + 9.2}{2} = 7 \tag{15}$$

So, $y(\mathbf{x})$ jumps from 5.5 to 7 when $x_2 = 0.4$ and x_1 moves from 0.1 to $0.1 + \delta$. Other jump discontinuities can be analyzed in a similar way.

IV. Conclusions

In this paper, the continuities and discontinuities of T1 and IT2 FLSs have been defined and investigated. Conditions under which an FLS gives a continuous/discontinous input-output mapping were derived. These results should be very useful in traditional fuzzy logic modeling and control, where usually a continuous input-output mapping is desired, and in hybrid and switched systems modeling and control, where a discontinuous input-output mapping is needed. Our future research includes how to apply discontinuous FLSs to hybrid and switched systems modeling and control.

Appendix: Proof of Theorem 1

Consider an arbitrary small number $\epsilon > 0$, and an arbitrary point $\mathbf{c} = (c_1, c_2, ..., c_M)$ in the input domain of $f(\mathbf{x})$. Because $g(\mathbf{x})$ is a continuous function of $\mathbf{x}$, we can always find a $\delta_1 > 0$ such that when $\max_m |x_m - c_m| < \delta_1$, $|g(\mathbf{x}) - g(\mathbf{c})| < \epsilon/3$. Since $f(\mathbf{x})$ universally approximates $g(\mathbf{x})$, we always have $|g(\mathbf{c}) - f(\mathbf{c})| < \epsilon/3$ for $\forall \mathbf{c}$, and we can find $\delta_2 > 0$ such that $|g(\mathbf{x}) - f(\mathbf{c})| < \epsilon/3$ when $\max_m |x_m - c_m| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. Then, when $\max_m |x_m - c_m| < \delta$,

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{c})| =& |f(\mathbf{x}) - f(\mathbf{c}) + g(\mathbf{x}) - g(\mathbf{x}) + g(\mathbf{c}) - g(\mathbf{c})| \\ \leqslant& |f(\mathbf{x}) - g(\mathbf{x})| + |g(\mathbf{c}) - f(\mathbf{c})| \\ &+ |g(\mathbf{x}) - g(\mathbf{c})| \\ <& \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned} \tag{16}$$

i.e., $f(\mathbf{x})$ is continuous at $\mathbf{c}$. Since $\mathbf{c}$ is an arbitrary point in the input domain of $f(\mathbf{x})$, $f(\mathbf{x})$ must be continuous in its entire input domain.

TABLE VII

FIRING INTERVALS OF THE IT2 FLSS SHOWN IN THE MIDDLE COLUMN OF FIG. 5 WHEN $x_1 = 0.1$ AND $x_2 = 0.4$.

Firing interval of antecedents		Fired rules	
x_1 domain	x_2 domain	Firing interval	Rule consequent
$[\mu_{\underline{X}_{11}}(0.1), \mu_{\overline{X}_{11}}(0.1)] = [0,0]$	$[\mu_{\underline{X}_{21}}(0.4), \mu_{\overline{X}_{11}}(0.4)] = [0,0]$	$[6/7, 9/10] \times [0, 3/5] = [0, .54]$	$[4.8, 5.2]$
$[\mu_{\underline{X}_{12}}(0.1), \mu_{\overline{X}_{12}}(0.1)] = [6/7, 9/10]$	$[\mu_{\underline{X}_{22}}(0.4), \mu_{\overline{X}_{12}}(0.4)] = [0, 3/5]$	$[6/7, 9/10] \times [0, 1/3] = [0, .3]$	$[5.8, 6.2]$
$[\mu_{\underline{X}_{13}}(0.1), \mu_{\overline{X}_{13}}(0.1)] = [0,0]$	$[\mu_{\underline{X}_{23}}(0.4), \mu_{\overline{X}_{13}}(0.4)] = [0, 1/3]$		

TABLE VIII

FIRING INTERVALS OF THE IT2 FLSS SHOWN IN THE MIDDLE COLUMN OF FIG. 5 WHEN $x_1 = 0.1 + \delta$ AND $x_2 = 0.4$.

Firing interval of antecedents		Fired rules	
x_1 domain	x_2 domain	Firing interval	Rule consequent
$[\mu_{\underline{X}_{11}}(0.1+\delta), \mu_{\overline{X}_{11}}(0.1+\delta)] = [0,0]$	$[\mu_{\underline{X}_{21}}(0.4), \mu_{\overline{X}_{11}}(0.4)] = [0,0]$	$[0, .54 - .6\delta]$	$[4.8, 5.2]$
$[\mu_{\underline{X}_{12}}(0.1+\delta), \mu_{\overline{X}_{12}}(0.1+\delta)] = [(.6-\delta)/.7, .9-\delta]$	$[\mu_{\underline{X}_{22}}(0.4), \mu_{\overline{X}_{12}}(0.4)] = [0, 3/5]$	$[0, .3 - \delta/3]$	$[5.8, 6.2]$
$[\mu_{\underline{X}_{13}}(0.1+\delta), \mu_{\overline{X}_{13}}(0.1+\delta)] = [0, \delta/.9]$	$[\mu_{\underline{X}_{23}}(0.4), \mu_{\overline{X}_{13}}(0.4)] = [0, 1/3]$	$[0, \delta/1.5]$	$[7.8, 8.2]$
		$[0, \delta/2.7]$	$[8.8, 9.2]$

A. *Proof of Theorem 2*

We need to prove that $y(\mathbf{x})$ in (1) is continuous at $\mathbf{c} = (c_1, c_2, ..., c_M)$. The firing level of the k^{th} rule, f_k, is computed by (2). Since each $\mu_{X_{m,n_{mn}}}(x_m)$ is a continuous function of x_m, and both product and minimum t-norms are continuous functions, f_k must be continuous at $\mathbf{c}$. Consequently, $\sum_{k=1}^K f_k y_k$ and $\sum_{k=1}^K f_k$ are also continuous at $\mathbf{c}$. As a result, $y(\mathbf{x})$ in (1) is continuous at $\mathbf{c}$ if and only if $\sum_{k=1}^K f_k \neq 0$, which holds if and only if $\max\limits_{n=1,2,...,N_m} \mu_{X_{mn}}(c_m) > 0$ for $\forall m = 1, 2, ...M$, i.e., every c_m is covered by some continuous T1 FSs.

B. *Proof of Theorem 3*

Follow the same line of reasoning in the proof of Theorem 2, $y(\mathbf{x})$ in (1) has a gap discontinuity at $\mathbf{c}$ if and only if $\sum_{k=1}^K f_k = 0$, which holds if and only if there exists c_m such that $\max\limits_{n=1,2,...,N_m} \mu_{X_{mn}}(c_m) = 0$, i.e., c_m is not covered by any continuous T1 FS in its domain.

C. *Proof of Theorem 4*

To show $y(\mathbf{x})$ has a gap discontinuity at $\mathbf{c}$ is equivalent to showing that at least one of $y_l(\mathbf{x})$ and $y_r(\mathbf{x})$ has a gap discontinuity at $\mathbf{c}$, i.e., $y(\mathbf{x})$ is undefined as long as at least one of $y_l(\mathbf{x})$ and $y_r(\mathbf{x})$ is undefined. We will show that $y_l(\mathbf{x})$ has a gap discontinuity at $\mathbf{c}$ if and only if $\exists c_m$ such that $\max\limits_{n=1,2,...,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$, i.e., there exist at least one c_m not covered by the UMFs. The condition is also true for $y_r(\mathbf{x})$. Since its proof is very similar to that for $y_l(\mathbf{x})$, it is left to the reader as an exercise.

Consider the sufficiency first. When $\exists c_m$ such that $\max\limits_{n=1,2,...,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$, all $\underline{f}_k$ and $\overline{f}_k$, computed by (6) and (7), equal 0; hence, the numerator of (4) is 0. Consequently, $y_l(\mathbf{x})$ is undefined at $\mathbf{c}$, i.e., $y_l(\mathbf{x})$ has a gap discontinuity at $\mathbf{c}$.

Next consider the necessity. When $\max\limits_{n=1,2,...,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0$ for $\forall m = 1, 2, ..., M$, there are $K' \geqslant 1$ rules whose $\overline{f}_k > 0$, and $y_l(\mathbf{x})$ is computed by (17), which is defined in this case; so, $y_l(\mathbf{x})$ does not have a gap discontinuity at $\mathbf{c}$. Consequently, to have a gap discontinuity at $\mathbf{c}$, there must $\exists c_m$ such that $\max\limits_{n=1,2,...,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$.

D. *Proof of Theorem 5*

A lemma on the sufficient condition of a continuous IT2 FLS is given first. It will be used in the proof of Theorem 5.

Lemma 2: The IT2 FLS is continuous at $\mathbf{c}$ if $\max\limits_{n=1,2,...,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall m = 1, 2, ...M$, i.e., every c_m is covered by some continuous LMFs. □

Proof: To prove $y(\mathbf{x})$ defined in (3) is continuous at $\mathbf{c}$ we need to show that both $y_l(\mathbf{x})$ and $y_r(\mathbf{x})$ are continuous at $\mathbf{c}$. We only show that for $y_l(\mathbf{x})$. The proof for $y_r(\mathbf{x})$ is very similar and hence left to the reader as an exercise.

It has been shown [9], [25] that $y_l(\mathbf{x})$ can be written as:

$$y_l(\mathbf{x}) = \min_{k' \in [1, K']} \frac{\sum_{k=1}^{k'} \overline{f}_k \underline{y}_k + \sum_{k=k'+1}^{K'} \underline{f}_k \underline{y}_k}{\sum_{k=1}^{k'} \overline{f}_k + \sum_{k=k'+1}^{K'} \underline{f}_k} \tag{17}$$

where K' is the number of fired rules, i.e., those rules with $\overline{f}_k > 0$. So, to show $y_l(\mathbf{x})$ is continuous at $\mathbf{c}$ we only need to show each $\frac{\sum_{k=1}^{k'} \overline{f}_k \underline{y}_k + \sum_{k=k'+1}^{K'} \underline{f}_k \underline{y}_k}{\sum_{k=1}^{k'} \overline{f}_k + \sum_{k=k'+1}^{K'} \underline{f}_k}$ is continuous at $\mathbf{c}$, because the minimum of several continuous functions is still continuous. When $\max\limits_{n=1,2,...,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for $\forall m = 1, 2, ...M$, i.e., every c_m is covered by some continuous LMFs, all $\underline{f}_k$ and $\overline{f}_k$ are continuous at $\mathbf{c}$, and hence $\sum_{k=1}^{k'} \overline{f}_k + \sum_{k=k'+1}^{K'} \underline{f}_k > 0$; so, $y_l(\mathbf{x})$ is continuous at $\mathbf{c}$. □

Next we prove Theorem 5. Consider the sufficiency first. When there exists an m such that $\max\limits_{n=1,2,...,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, all firing levels $\underline{f}_k$ are zero for $\mathbf{c}$. Consider $y_l(\mathbf{x})$ in (17). Its minimum is achieved when $k = 1$, i.e., $y_l(\mathbf{x}) = \underline{y}_1 = \min\limits_{k=1,...,K'} \underline{y}_k$.

When any $c_{m'}$ $(m' \neq m)$ changes to $c_{m'} + \delta$, because $\max\limits_{n=1,2,...,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, all firing levels $\underline{f}_k$ are still zero; hence, $y_l(\mathbf{x})$ still equals the minimum $\underline{y}_k$ of all firing rules. Clearly, if the minimum $\underline{y}_k$ of all firing rules changes as any $c_{m'}$ $(m' \neq m)$ changes to $c_{m'} + \delta$, then there is a jump in $y_l(\mathbf{x})$, and hence the input-output mapping has a jump discontinuity at $\mathbf{c}$. Similarly, if the maximum $\overline{y}_k$ of all firing rules changes as any $c_{m'}$ ($m' \neq m$) changes to $c_{m'} + \delta$, then there is a jump in $y_r(\mathbf{x})$, and hence the input-output mapping has a jump discontinuity at $\mathbf{c}$.

Next consider the necessity. $y(\mathbf{c})$ can only have three cases:

1) There exists c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) = 0$.
2) $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0$ and $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) > 0$ for all c_m.
3) $\max_{n=1,2,\ldots,N_m} \mu_{\overline{X}_{mn}}(c_m) > 0$ for all c_m, but there exists c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$.

The first case has a gap discontinuity at $\mathbf{c}$, according to Theorem 4. The second case is continuous at $\mathbf{c}$, according to Lemma 2. Then, a jump discontinuity can only happen in the third case, as indicated by the first two criteria in Theorem 5. When there exists c_m such that $\max_{n=1,2,\ldots,N_m} \mu_{\underline{X}_{mn}}(c_m) = 0$, the lower bounds of all fired rules are 0. In this case, $y_l(\mathbf{c})$ equals the minimum $\underline{y}_k$ of all fired rules, and $y_r(\mathbf{c})$ equals the maximum $\overline{y}_k$ of all fired rules. So, if Criterion 3 of Theorem 5 is not satisfied, then both $y_l(\mathbf{c})$ and $y_r(\mathbf{c})$ are not changed when $\mathbf{c}$ changes; hence, there is no jump discontinuity at $\mathbf{c}$. In other words, to have a jump discontinuity at $\mathbf{c}$, the minimum $\underline{y}_k$ and/or maximum $\overline{y}_k$ of all fired rules must be different as $\mathbf{c}$ changes.

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