Enhanced Interval Approach for Encoding Words Into Interval Type-2 Fuzzy Sets and Its Convergence Analysis

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***Abstract*—Construction of interval type-2 fuzzy set models is the first step in the perceptual computer, which is an implementation of computing with words. The interval approach (IA) has, so far, been the only systematic method to construct such models from data intervals that are collected from a survey. However, as pointed out in this paper, it has some limitations, and its performance can be further improved. This paper proposes an enhanced interval approach (EIA) and demonstrates its performance on data that are collected from a web survey. The data part of the EIA has more strict and reasonable tests than the IA, and the fuzzy set part of the EIA has an improved procedure to compute the lower membership function. We also perform a convergence analysis to answer two important questions: 1) Does the output interval type-2 fuzzy set from the EIA converge to a stable model as increasingly more data intervals are collected, and 2) if it converges, then how many data intervals are needed before the resulting interval type-2 fuzzy set is sufficiently similar to the model obtained from infinitely many data intervals? We show that the EIA converges in a mean-square sense, and generally, 30 data intervals seem to be a good compromise between cost and accuracy.**

***Index Terms*—Computing with words (CWW), convergence analysis, enhanced interval approach (EIA), interval approach (IA), interval type-2 fuzzy sets (IT2 FSs), perceptual computing.**

I. Introduction

ZADEH coined the phrase "*computing with words*" (CWW) [52], [53], which is [53] "*a methodology in which the objects of computation are words and propositions drawn from a natural language*." Words in the CWW paradigm may be modeled by type-1 fuzzy sets (T1 FSs) [50] or their extension, i.e., interval type-2 fuzzy sets (IT2 FSs) [21], [51]. CWW using T1 FSs has been studied by many researchers, e.g., [4], [10], [11], [13], [14], [31], [33], [34], [37], [49], and [52]–[54]; however, since IT2 FSs can model both interpersonal and intrapersonal uncertainties [23], [24], and T1 FSs are a special case of IT2 FSs, in this paper, we focus on CWW using IT2 FSs. By interpersonal uncertainties, we mean the variations in the understanding of a words between people, and by intrapersonal uncertainties, we mean the variation in one person's understanding of a word over time. These linguistic uncertainties are at the heart of the CWW paradigm. CWW using IT2 FSs have also been studied by several researchers [30], [43].

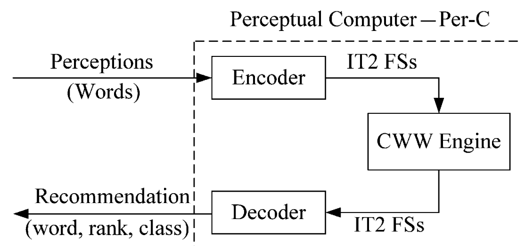

Fig. 1. Conceptual structure of the Per-C.

A specific architecture, which is proposed in [22] and elaborated upon in [29] and [38] for making subjective judgments by CWW, is shown in Fig. 1. It is called a *Perceptual Computer*—Per-C for short. In Fig. 1, the *encoder* transforms linguistic perceptions into IT2 FSs that activate a *CWW engine*. The CWW engine performs operations on the IT2 FSs. The *decoder* maps the output of the CWW engine into a recommendation, which can be a word, rank, or class.

As shown in Fig. 1, the first step in the Per-C is to transform words into IT2 FSs, i.e., the encoding problem. Liu and Mendel have proposed an *Interval Approach* (IA) [17] to synthesize an IT2 FS model for a word, in which, interval endpoint data about a word are collected from a group of subjects (the subjects are asked: *On a scale of 0–10, what are the endpoints of an interval that you associate with the word ____?*); each subject's data interval is mapped into a T1 FS; the latter is interpreted as an embedded T1 FS [21] of an IT2 FS, and an IT2 FS mathematical model is obtained for the word from these T1 FSs. However, as pointed out in Section II-C, there are some limitations to the IA.

In this paper, we propose an enhanced interval approach (EIA) to overcome these limitations. Additionally, we also design experiments to demonstrate that the IT2 FS output from the EIA converges to a stable model as more and more data intervals are collected. We also find empirically the number of data intervals that one should collect before the IT2 FS is sufficiently similar to its underlying (unknown) reference model, which is obtained when infinitely many data intervals are collected.

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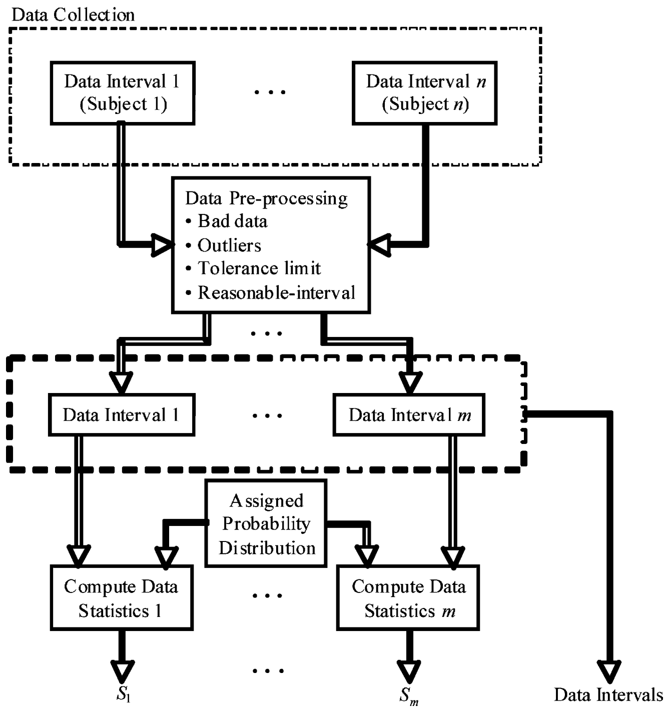


Fig. 2. Data part of the IA [17]. Note that the output statistics feed into the fuzzy set part of the IA in Fig. 3.

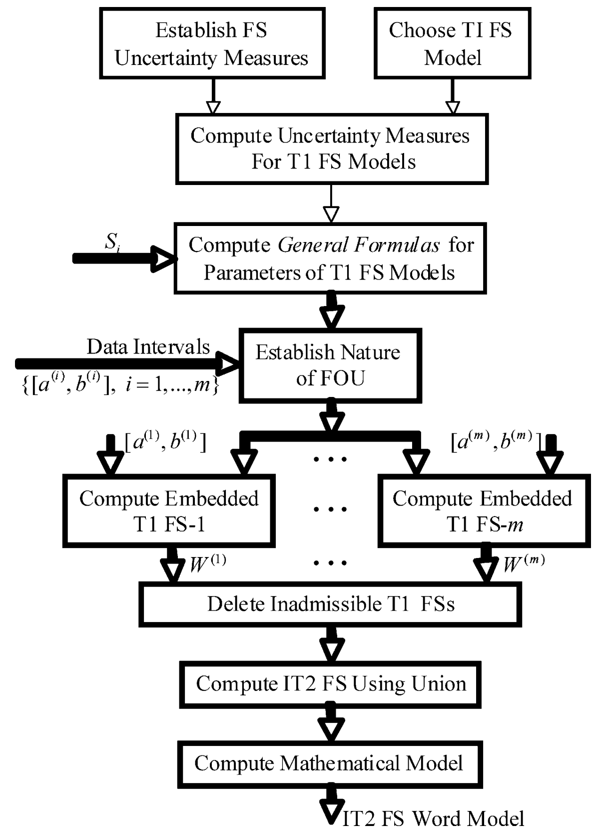


Fig. 3. FS part of the IA [17].

The rest of this paper is organized as follows. Section II introduces the IA and points out its limitations. Section III presents the EIA and an example to demonstrate it. Section IV studies the convergence of the EIA. Finally, Section V draws conclusions. Background knowledge about IT2 FSs is given in the Appendix.

II. Interval Approach

Liu and Mendel [17] proposed an IA to construct IT2 FS word models from interval endpoints data. It is briefly introduced in this section. For detailed explanations of the method and its associated formulas, see [17] or [29].

A. Interval Approach Algorithm

The IA consists of two parts: the *data part* (see Fig. 2) and the *FS part* (see Fig. 3). In the data part, for each word in an application-dependent encoding vocabulary, a group of n subjects are asked the following question:

On a scale of 0–10, what are the endpoints of an interval that you associate with the word ____?

n data intervals $[a^{(i)}, b^{(i)}]$ are then collected from these subjects. They are then preprocessed by the following four steps,[1] as shown in Fig. 2:

1) *Bad data processing*: Only data with $0 \leqslant a^{(i)} < b^{(i)} \leqslant 10$ are accepted; others are rejected. This step reduces n interval endpoints to n' interval endpoints.
2) *Outlier processing*: Box and Whisker tests [35] are performed on the remaining n' $a^{(i)}$, $b^{(i)}$, and $L^{(i)} = b^{(i)} - a^{(i)}$ simultaneously, i.e., only intervals satisfying

$$a^{(i)} \in [Q_a(.25) - 1.5IQR_a, Q_a(.75) + 1.5IQR_a]$$
$$b^{(i)} \in [Q_b(.25) - 1.5IQR_b, Q_b(.75) + 1.5IQR_b]$$
$$L^{(i)} \in [Q_L(.25) - 1.5IQR_L, Q_L(.75) + 1.5IQR_L]$$

are kept, where Q_a (Q_b, Q_L) and IQR_a (IQR_b, IQR_L) are the quartiles and interquartile ranges for the left (right) endpoints and interval length.

After outlier processing, there will be $m' \leqslant n'$ remaining data intervals for which the following data statistics are then computed: m_a, σ_a (sample mean and standard deviation (std) of the m' left endpoints), m_b, σ_b (sample

[1]There are four steps in the data part of the IA as well as EIA: 1) *bad data processing*, where very obvious bad data are removed, and no statistics are used, 2) *outlier processing*, where simple statistics are used, but no probability distribution assumption about the data is made; 3) *tolerance limit processing*, where the data are assumed to have Gaussian distribution, and only a certain percentage of the responses are kept; and 4) *reasonable-interval processing*, where the data are assumed to have Gaussian distribution, and the assumption "words must mean similar things to different people" is used. Note that these four steps are increasingly demanding, requiring more assumptions and more understanding about the nature of the data. Each step relies on its previous steps: Bad data processing helps improve the accuracy of the quartile and interquartile distance computation in outlier processing; outlier processing removes some obvious outliers and helps improve the mean and std computation in tolerance limit processing; tolerance limit processing further prepares the data for reasonable-interval processing under the same Gaussian distribution assumption. The goal of tolerance limit processing is to remove some minority responses so that the mean and std of the data can be more accurately computed, which are used in reasonable-interval processing.

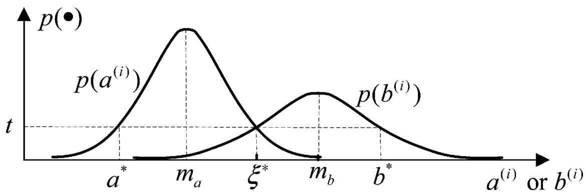


Fig. 4. Reasonable interval tests. For IA, reasonable intervals must have $a^{(i)} < \xi^* < b^{(i)}$. For EIA, reasonable intervals must have $a^* < a^{(i)} < \xi^* < b^{(i)} < b^*$.

mean and std of the m' right endpoints), and m_L, σ_L (sample mean and std of the lengths of the m' intervals).

3) *Tolerance limit processing*: Tolerance limit processing[2] is performed on the remaining m' $a^{(i)}$, $b^{(i)}$, and $L^{(i)}$ simultaneously, and only intervals satisfying

$$a^{(i)} \in [m_a - k\sigma_a, m_a + k\sigma_a]$$
$$b^{(i)} \in [m_b - k\sigma_b, m_b + k\sigma_b]$$
$$L^{(i)} \in [m_L - k\sigma_L, m_L + k\sigma_L]$$

are kept, where the *tolerance factor* k is determined so that one can assert with $100(1-\gamma)\%$ confidence that the given limits contain at least the proportion $1-\alpha$ of the measurements [35]. Note that we assume the data interval endpoints are approximately normal. k can be found from a lookup table, e.g., [35, Table A.7].

After tolerance limit processing, there will be $m'' \leqslant m'$ remaining data intervals ($1 \leqslant m'' \leqslant n$), and the following data statistics are then recomputed: m_a, σ_a, (sample mean and std of the m'' left endpoints), and m_b, σ_b (sample mean and std of the m'' right endpoints).

4) *Reasonable-interval processing*: In this step, intervals that have little overlap[3] with others are removed. It is assumed that both the left endpoints and the right endpoints obey a Gaussian distribution, and each reasonable interval must contain the point that best separates the two Gaussian distributions (see Fig. 4). To do this, one finds one of the values

$$\xi^* = \{(m_b\sigma_a^2 - m_a\sigma_b^2) \pm \sigma_a\sigma_b[(m_a - m_b)^2 + 2(\sigma_a^2 - \sigma_b^2)\ln(\sigma_a/\sigma_b)]^{1/2}\}/(\sigma_a^2 - \sigma_b^2) \quad (1)$$

such that

$$m_a \leqslant \xi^* \leqslant m_b$$

and then only keeps intervals such that

$$a^{(i)} < \xi^* < b^{(i)}.$$

This step reduces m'' interval endpoints to m interval endpoints.

Finally, a uniform distribution is assigned to each of the remaining m intervals $[a^{(i)}, b^{(i)}]$, and its mean and std are computed as follows:

$$m^{(i)} = \frac{a^{(i)} + b^{(i)}}{2} \quad (2)$$

$$\sigma^{(i)} = \frac{b^{(i)} - a^{(i)}}{\sqrt{12}}. \quad (3)$$

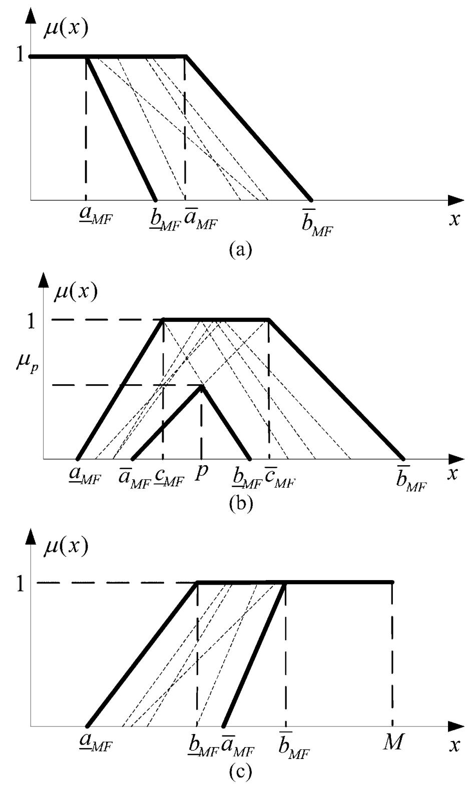


Fig. 5. Examples of the union of (dashed) T1 MFs. The heavy lines are the LMFs and UMFs for the FOU. (a) Left shoulder, (b) interior FOU, and (c) right shoulder.

In the FS part (see Fig. 3), the nature of the footprint of uncertainty (FOU) (interior FOU, left shoulder or right shoulder, see Fig. 5) is determined first, and then, each of the word's data intervals is individually mapped into its respective T1 interior, left-shoulder or right-shoulder membership function (MF), after which the lower MF (LMF) and upper MF (UMF) of the IT2 FS are computed.

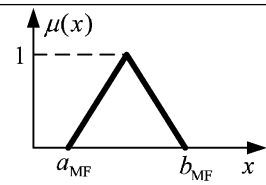
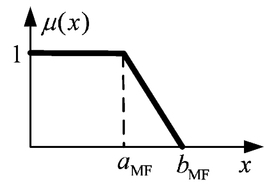
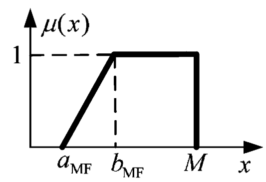
There can be different methods to map a data interval into a T1 FS. In the IA, this is achieved by equating $m^{(i)}$ in (2) and $\sigma^{(i)}$ in (3) to the mean and std of a T1 FS, which is defined as

$$m_{\mathrm{MF}} = \frac{\int_{a_{\mathrm{MF}}}^{b_{\mathrm{MF}}} x\mu_{\mathrm{MF}}(x)dx}{\int_{a_{\mathrm{MF}}}^{b_{\mathrm{MF}}} \mu_{\mathrm{MF}}(x)dx} \quad (4)$$

[2]In this paper, we use *tolerance interval* instead of *confidence interval*. A tolerance interval is a statistical interval within which, with some probability, a specified proportion of a population falls. It differs from a confidence interval in that the confidence interval bounds a single-valued population parameter (the mean or the variance, for example) with some confidence, while the bounds of a tolerance interval are a range of possible data values that represent a specified ratio of the population. In simpler terms, the confidence interval estimates the range in which a population parameter falls, whereas the tolerance interval estimates the range which should contain a certain percentage of each individual measurement in the population. Clearly, tolerance interval is more appropriate for our application, because we want to keep a certain percentage of the responses.

[3]Overlap of intervals is associated with "words must mean similar things to different people," or else, effective communication is not possible.

TABLE I
TRANSFORMATION OF THE UNIFORMLY DISTRIBUTED DATA INTERVAL $[a, b]$ INTO THE PARAMETERS a_{MF} AND b_{MF} OF A T1 FS [17]

Name	MF	m_{MF} and σ_{MF}	Transformation
Symmetric triangle (Interior MF)		$m_{\mathrm{MF}} = \frac{a_{\mathrm{MF}}+b_{\mathrm{MF}}}{2}$ $\sigma_{\mathrm{MF}} = \frac{b_{\mathrm{MF}}-a_{\mathrm{MF}}}{2\sqrt{6}}$	$a_{\mathrm{MF}} = \frac{a+b}{2} - \frac{b-a}{\sqrt{2}}$ $b_{\mathrm{MF}} = \frac{a+b}{2} + \frac{b-a}{\sqrt{2}}$
Left shoulder		$m_{\mathrm{MF}} = \frac{2a_{\mathrm{MF}}+b_{\mathrm{MF}}}{3}$ $\sigma_{\mathrm{MF}} = \frac{b_{\mathrm{MF}}-a_{\mathrm{MF}}}{3\sqrt{2}}$	$a_{\mathrm{MF}} = \frac{a+b}{2} - \frac{b-a}{\sqrt{6}}$ $b_{\mathrm{MF}} = \frac{a+b}{2} + \frac{\sqrt{6}(b-a)}{3}$
Right shoulder		$m_{\mathrm{MF}} = \frac{a_{\mathrm{MF}}+2b_{\mathrm{MF}}}{3}$ $\sigma_{\mathrm{MF}} = \frac{b_{\mathrm{MF}}-a_{\mathrm{MF}}}{3\sqrt{2}}$	$a_{\mathrm{MF}} = \frac{a+b}{2} - \frac{\sqrt{6}(b-a)}{3}$ $b_{\mathrm{MF}} = \frac{a+b}{2} + \frac{b-a}{\sqrt{6}}$

$$\sigma_{\mathrm{MF}} = \frac{\int_{a_{\mathrm{MF}}}^{b_{\mathrm{MF}}} (x - m_{\mathrm{MF}})^2 \mu_{\mathrm{MF}}(x) dx}{\int_{a_{\mathrm{MF}}}^{b_{\mathrm{MF}}} \mu_{\mathrm{MF}}(x) dx} \tag{5}$$

where a_{MF} and b_{MF} are the parameters of the MFs depicted in the figures in Table I. For the simple T1 FSs used in this paper, m_A and σ_A have closed-form expressions, as shown in the third column of Table I. The transformations between the data interval $[a, b]$ and the FS parameters $\{a_{\mathrm{MF}}, b_{\mathrm{MF}}\}$ are given in the last column of Table I.

The FS part of the IA consists of the following steps, as shown in Fig. 3.

1) Compute the means of the remaining m left and right endpoints:

$$m_l = \frac{1}{m} \sum_{i=1}^{m} a^{(i)} \tag{6}$$

$$m_r = \frac{1}{m} \sum_{i=1}^{m} b^{(i)} \tag{7}$$

2) Define[4]

$$c^{(i)} \equiv b^{(i)} - 5.831 a^{(i)}, \quad i = 1, 2, \ldots, m \tag{8}$$

$$d^{(i)} \equiv b^{(i)} - 0.171 a^{(i)} - 8.29, \quad i = 1, 2, \ldots, m \tag{9}$$

and compute s_c as the std of $c^{(i)}$, and s_d as the std of $d^{(i)}$.

3) Classify the FOU according to the following rules (see Fig. 6):

a) If $m_r \leqslant 5.831 m_l - t_{\alpha,m-1}\frac{s_c}{\sqrt{m}}$, $m_r \leqslant 0.171 m_l + 8.29 - t_{\alpha,m-1}\frac{s_d}{\sqrt{m}}$, and $m_r \geqslant m_l$, then FOU is interior.

b) Otherwise, if $m_r > 5.831 m_l - t_{\alpha,m-1}\frac{s_c}{\sqrt{m}}$ and $m_r < 0.171 m_l + 8.29 - t_{\alpha,m-1}\frac{s_d}{\sqrt{m}}$, then FOU is a left shoulder.

c) Otherwise, if $m_r < 5.831 m_l - t_{\alpha,m-1}\frac{s_c}{\sqrt{m}}$ and $m_r > 0.171 m_l + 8.29 - t_{\alpha,m-1}\frac{s_d}{\sqrt{m}}$, then FOU is a right shoulder.

d) Otherwise, there is no FOU.

Note that $t_{\alpha,m-1}$ is a parameter used in a one-tailed test [35] and can be found from a lookup table. The classification diagram for Step 3 is depicted in Fig. 6.

4) Map each of the m data intervals $[a^{(i)}, b^{(i)}]$ into the corresponding MF parameters $\{a_{\mathrm{MF}}^{(i)}, b_{\mathrm{MF}}^{(i)}\}$ using the formulas in the last column of Table I. This results in m embedded T1 FSs.

5) Delete the embedded T1 FSs that have $b_{\mathrm{MF}}^{(i)} > 10$ and/or $b_{\mathrm{MF}}^{(i)} < a_{\mathrm{MF}}^{(i)}$. This step reduces the number of embedded T1 FSs from m to m^*.

6) Construct an IT2 FS model from the m^* embedded T1 FSs. Compute $\{\underline{a}_{\mathrm{MF}}, \underline{c}_{\mathrm{MF}}, \overline{c}_{\mathrm{MF}}, \overline{b}_{\mathrm{MF}}; \overline{a}_{\mathrm{MF}}, \underline{b}_{\mathrm{MF}}, p, \mu_p\}$ if it is an interior FOU [see Fig. 5(b)]. Compute $\{\underline{a}_{\mathrm{MF}}, \underline{b}_{\mathrm{MF}}; \overline{a}_{\mathrm{MF}}, \overline{b}_{\mathrm{MF}}\}$ if it is a left or right shoulder [see Fig. 5(a) and (c)], as

$$\underline{a}_{\mathrm{MF}} = \min_{i=1,\ldots,m^*} \{a_{\mathrm{MF}}^{(i)}\} \tag{10}$$

$$\overline{a}_{\mathrm{MF}} = \max_{i=1,\ldots,m^*} \{a_{\mathrm{MF}}^{(i)}\} \tag{11}$$

$$\underline{b}_{\mathrm{MF}} = \min_{i=1,\ldots,m^*} \{b_{\mathrm{MF}}^{(i)}\} \tag{12}$$

[4] An interior FOU is admissible if and only if $a_{\mathrm{MF}}^{(i)} \geqslant 0$ and $b_{\mathrm{MF}}^{(i)} \leqslant 10$ for $\forall i = 1, \ldots, m$. By using the formulas for symmetric triangle in the last column of Table I, these inequalities are equivalent to $b^{(i)} - 5.831 a^{(i)} \leqslant 0$ and $b^{(i)} - 0.171 a^{(i)} - 8.29 \leqslant 0$. That is how $c^{(i)}$ and $d^{(i)}$ originate. Detailed derivations are given in [17].

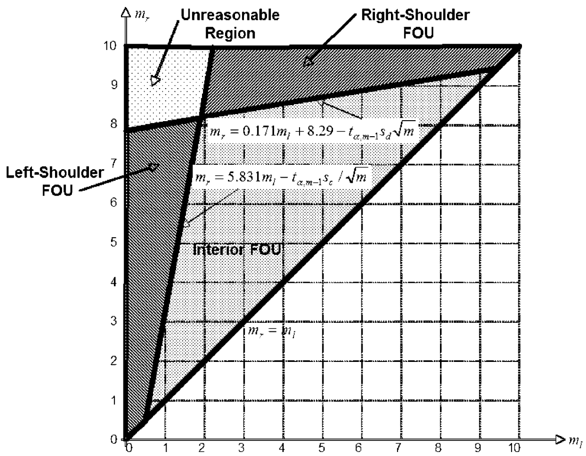


Fig. 6. Classification diagram for IA and EIA.

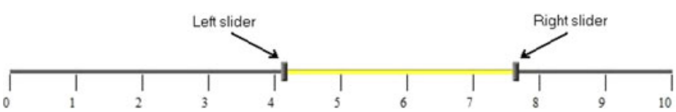


Fig. 7. Double-ended slider used to collect intervals.

$$\overline{b}_{\mathrm{MF}} = \max_{i=1,\ldots,m^*}\{b_{\mathrm{MF}}^{(i)}\} \tag{13}$$

$$c_{\mathrm{MF}}^{(i)} = \frac{a_{\mathrm{MF}}^{(i)} + b_{\mathrm{MF}}^{(i)}}{2} \tag{14}$$

$$\underline{c}_{\mathrm{MF}} = \min_{i=1,\ldots,m^*}\{c_{\mathrm{MF}}^{(i)}\} \tag{15}$$

$$\overline{c}_{\mathrm{MF}} = \max_{i=1,\ldots,m^*}\{c_{\mathrm{MF}}^{(i)}\} \tag{16}$$

$$p = \frac{\underline{b}_{\mathrm{MF}}(\overline{c}_{\mathrm{MF}} - \overline{a}_{\mathrm{MF}}) + \overline{a}_{\mathrm{MF}}(\underline{b}_{\mathrm{MF}} - \underline{c}_{\mathrm{MF}})}{(\overline{c}_{\mathrm{MF}} - \overline{a}_{\mathrm{MF}}) + (\underline{b}_{\mathrm{MF}} - \underline{c}_{\mathrm{MF}})} \tag{17}$$

$$\mu_p = \frac{\underline{b}_{\mathrm{MF}} - p}{\underline{b}_{\mathrm{MF}} - \underline{c}_{\mathrm{MF}}}. \tag{18}$$

A word that is modeled by an interior FOU has a UMF that is a trapezoid and an LMF that is a triangle, but in general, neither the trapezoid nor the triangle is symmetrical, as shown in Fig. 5(b). A word that is modeled as a left- or right-shoulder FOU has trapezoidal UMF and LMF; however, the legs of the respective two trapezoids are not necessarily parallel, as shown in Fig. 5(a) and (c).

B. Example

In 2009, the third author conducted an online survey where participants were invited to give the interval which best describes a word on the interval scale of 0 to 10 using a pair of sliders as depicted in Fig. 7. The users were free to enter any value (subject to the slider bar resolution) between 0 and 10 for each endpoint, with the condition that the left endpoint must be less than or equal to the right endpoint. This interval data was gathered for a set of 32 words, which can be grouped into three classes: small-sounding words (*little, low amount, somewhat small, a smidgen, none to very little, very small, very little, teeny-weeny, small amount,* and *tiny*), medium-sounding words (*fair amount, modest amount, moderate amount, medium, good amount, a bit, some to moderate,* and *some*), and large-sounding words (*sizeable, large, quite a bit, humongous amount, very large, extreme amount, considerable amount, a lot, very sizeable, high amount, maximum amount, very high amount*, and *substantial amount*). Generally, the words were presented in a randomized manner; however, because many users did not finish the survey for all 32 words, words that had fewer responses were presented first to new respondents. Eventually, 175 responses were collected for each word.

When the IA was applied to this dataset, the remaining numbers of data intervals after each processing stage are shown in Table II. The resulting IT2 FSs are the ones that are shown in Fig. 8, where the m^* embedded T1 FSs for each IT2 FS are also shown. The number in the title of each subfigure is the area of corresponding FOU. Observe that there are 10 left shoulders, nine interior FOUs, and 13 right shoulders.

C. Limitations of the Interval Approach

Observe from Fig. 8 that, generally, the FOUs seem too fat and too wide. We believe there are two reasons for this; the first of which is due to the nature of the online survey method. The lack of contact between the people conducting the survey and the participants means there was less risk of influencing the data being collected; however, it also means there was no opportunity to explain the survey face to face or to answer questions about the survey, which may have led to some participants not understanding the survey. The data collection method, a two tailed slider, meant that participants could enter data simply and intuitively; however, it may not have been clear to all participants that both sliders could be changed. We observed a small number of users who entered a 0 value for the left endpoint for every single word. This data were not removed manually because we believed it should be captured by the preprocessing stage within the data part of the IA automatically. The second reason for the ungainly FOUs is that there may be limitations in the data part of IA to clean up the data. This is where the IA can be enhanced, as described in Section III.

Observe, also, from Fig. 8 that the LMFs of the interior FOUs obtained from the IA usually have very small height, i.e., μ_p in (18) may be very close to 0 (e.g., see the FOU for *Medium*). When examining (18) more carefully, we found that it only considers the case when the LMF of an interior FOU is completely determined by the two embedded T1 FSs that also determine the UMF, and these two embedded T1 FSs must form the perfect triangle $(\overline{a}_{\mathrm{MF}}, p, \underline{b}_{\mathrm{MF}})$, as shown in Fig. 5(b). Consider the example in Fig. 9(a), where there does not exist a perfect triangle for the LMF because the intersection of the three embedded T1 FSs is a quadrilateral indicated by the thick dashed lines. The IA artificially constructs two lines: one connecting $(\overline{a}_{\mathrm{MF}}, 0)$ and $(\overline{c}_{\mathrm{MF}}, 1)$ and the other one connecting $(\underline{b}_{\mathrm{MF}}, 0)$ and $(\underline{c}_{\mathrm{MF}}, 1)$, which is shown as the red-dashed lines in Fig. 9(b), and then finds (p, μ_p) as the intersection of these two lines. We believe it is more reasonable to find (p, μ_p) from the intersections of existing embedded T1 FSs, as shown in Fig. 9(c). More

TABLE II
REMAINING NUMBERS OF DATA INTERVALS AFTER EACH PROCESSING STAGE

Word	IA						EIA							
	Data Part					FS Part	Data Part							FS Part
	n	n'	m'	m''	m	m^*	n	n'	n''	m'	m^+	m''	m	m^*
Teeny-weeny	175	175	129	112	74	74	175	161	133	114	103	92	50	50
Tiny	175	174	125	111	90	90	175	165	126	119	107	95	65	65
None to very little	175	174	117	114	114	114	175	162	119	112	102	94	91	91
A smidgen	175	174	145	137	103	70	175	152	127	107	97	90	45	45
Very small	175	174	130	118	100	100	175	167	130	127	111	100	81	81
Very little	175	174	131	117	102	102	175	167	131	129	113	104	81	81
A bit	175	174	130	123	84	84	175	164	129	127	118	112	67	67
Little	175	174	134	126	91	91	175	166	134	130	119	113	76	76
Low amount	175	174	134	120	103	103	175	164	135	127	109	101	96	96
Small	175	174	138	128	108	108	175	167	138	137	123	118	99	99
Somewhat small	175	174	141	127	90	60	175	167	141	141	128	121	82	52
Some	175	174	158	146	96	67	175	162	161	146	124	117	47	43
Quite a bit	175	175	161	149	39	22	175	163	163	149	144	129	7	7
Modest amount	175	174	163	156	77	67	175	168	168	161	152	145	52	52
Some to moderate	175	174	162	149	117	96	175	164	161	157	136	129	91	90
Medium	175	174	135	129	126	126	175	165	135	134	120	117	117	117
Moderate amount	175	174	155	140	110	101	175	162	122	120	102	97	89	89
Fair amount	175	174	164	154	98	74	175	165	162	160	134	127	65	65
Good amount	175	174	161	146	91	49	175	162	161	156	142	130	41	40
Considerable amount	175	174	158	137	85	82	175	163	159	156	138	131	73	39
Sizeable	175	174	170	163	101	61	175	152	148	147	137	128	51	51
Substantial amount	175	175	171	163	117	89	175	158	155	144	128	120	77	77
Large	175	174	141	124	107	107	175	150	133	129	114	106	94	94
Very sizeable	175	174	156	144	118	88	175	155	143	129	112	106	75	75
A lot	175	174	137	126	113	113	175	164	134	125	109	103	91	91
High amount	175	174	137	122	96	96	175	163	138	136	118	113	86	86
Very large	175	174	113	106	105	105	175	154	120	115	103	99	87	87
Very high amount	175	174	120	117	117	117	175	161	120	118	114	111	111	111
Huge amount	175	174	119	114	114	114	175	155	115	114	106	101	99	99
Humongous amount	175	175	136	136	136	108	175	148	114	102	94	84	80	80
Extreme amount	175	174	121	118	118	118	175	147	118	111	101	90	83	83
Maximum amount	175	174	135	129	129	129	175	148	130	118	106	84	77	77

specifically, (p, μ_p) should be the lowest intersection of the left legs of existing embedded T1 FSs with the right legs. Comparing the two μ_p in Fig. 9(b) and (c), one can observe that the latter is larger, i.e., the latter approach gives an LMF with larger height. The EIA proposed in the next section adopts the latter approach. Additionally, some improvements to the preprocessing steps are made in the data part to clean the data.

III. ENHANCED INTERVAL APPROACH

The EIA is proposed in this section, and its performance is demonstrated using the same dataset in the previous section.

A. Enhanced Interval Approach Algorithm

The structure of the EIA is very similar to the IA. It again consists of a data part and an FS part, which are depicted in Figs. 2 and 3. The data part has the following steps.

1) *Bad data processing*: Only intervals with $0 \leqslant a^{(i)} < b^{(i)} \leqslant 10$ and $b^{(i)} - a^{(i)} < 10$ are accepted; others are rejected. This step reduces n interval endpoints to n' interval endpoints.

   Note that compared with the bad data processing step in the IA, here there is an extra requirement that $b^{(i)} - a^{(i)} < 10$ to remove intervals that span the entire range of $[0, 10]$.

2) *Outlier processing*: Box and Whisker tests are first performed on $a^{(i)}$ and $b^{(i)}$ and then on $L^{(i)} = b^{(i)} - a^{(i)}$, i.e., first, $Q_a(.25)$, $Q_a(.75)$, IQR_a, $Q_b(.25)$, $Q_b(.75)$, and IQR_b are computed based on the data from Step 1, and then, only intervals satisfying the following are kept:

$$a^{(i)} \in [Q_a(.25) - 1.5IQR_a, Q_a(.75) + 1.5IQR_a]$$
$$b^{(i)} \in [Q_b(.25) - 1.5IQR_b, Q_b(.75) + 1.5IQR_b].$$

   This step reduces n' interval endpoints to n'' interval endpoints. Then, $Q_L(.25)$, $Q_L(.75)$, and IQR_L are computed based on the remaining n'' intervals, and only intervals satisfying the following are kept:

$$L^{(i)} \in [Q_L(.25) - 1.5IQR_L, Q_L(.75) + 1.5IQR_L].$$

   This step reduces n'' interval endpoints to m' interval endpoints.

   Note that in the IA, these three tests are performed simultaneously. Here, the test on the length of the intervals is separated from the tests on the endpoints because outlier values of $a^{(i)}$ and $b^{(i)}$ can make IQR_L so large that $Q_L(.25) - 1.5IQR_L$ can be negative; hence, the IA Box and Whisker test on $L^{(i)}$ is not effective for removing short-length intervals, which contributes to a small LMF and a fat FOU. To address this problem, outlier values of $a^{(i)}$ and $b^{(i)}$ should be removed before testing $L^{(i)}$.

3) *Tolerance limit processing*: Tolerance limit processing on $a^{(i)}$ and $b^{(i)}$ is performed first, and then on $L^{(i)} = b^{(i)} - a^{(i)}$. For the former, only intervals satisfying the following are kept:

$$a^{(i)} \in [m_a - k\sigma_a, m_a + k\sigma_a] \quad (19)$$
$$b^{(i)} \in [m_b - k\sigma_b, m_b + k\sigma_b] \quad (20)$$

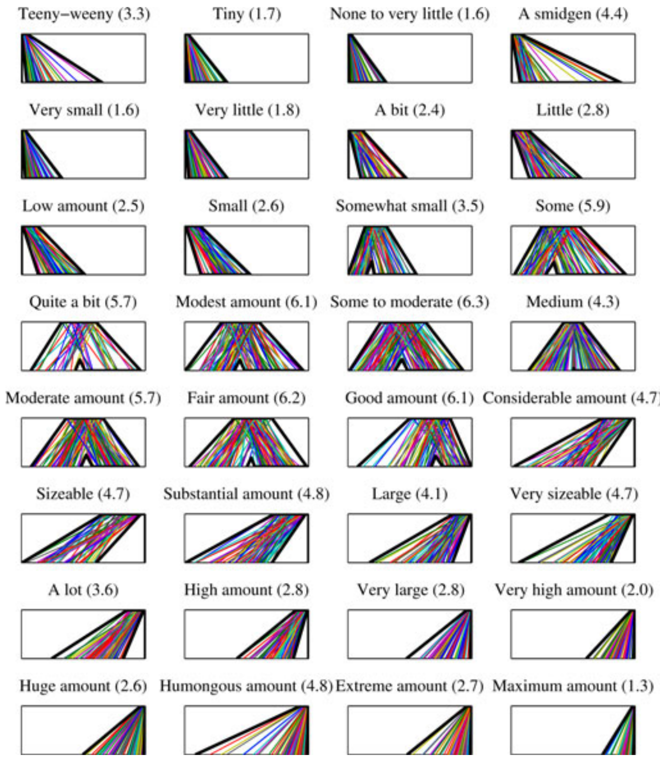


Fig. 8. IT2 FS word models obtained using the IA.

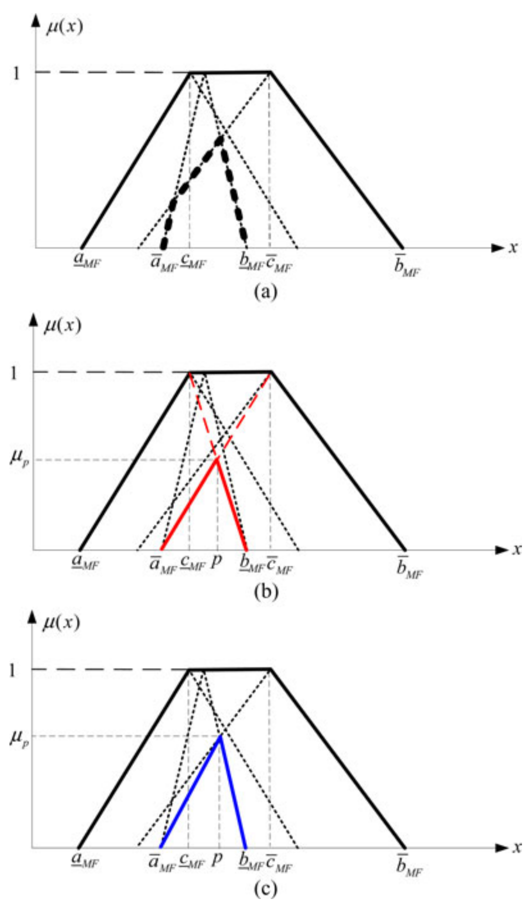


Fig. 9. (a) Case where the IA gives an LMF from the EIA. (b) LMF given by the IA. (c) LMF given by the EIA.

where k is determined such that one can assert with 95% confidence that the given limits contain at least 95% of the subject data intervals. Note that we assume the data interval endpoints are approximately normal. This step reduces m' interval endpoints to m^+ interval endpoints. m_L and σ_L are then computed based on the remaining data, and only intervals satisfying the following are kept:

$$L^{(i)} \in [m_L - k'\sigma_L, m_L + k'\sigma_L] \tag{21}$$

where

$$k' = \min(k_1, k_2, k_3) \tag{22}$$

in which k_1 is determined such that one can assert with 95% confidence that $[m_L - k_1\sigma_L, m_L + k_1\sigma_L]$ contains at least 95% of $L^{(i)}$, and

$$k_2 = m_L/\sigma_L \tag{23}$$

$$k_3 = (10 - m_L)/\sigma_L. \tag{24}$$

Equation (23) ensures that $m_L - k'\sigma_L \geqslant 0$, and (24) ensures that $m_L + k'\sigma_L \leqslant 10$ so that intervals with too small or too large $L^{(i)}$ can be removed. This step reduces m^+ interval endpoints to m'' interval endpoints.

Compared with the tolerance limit processing step in IA, two modifications have been introduced by us.

a) The test for $L^{(i)}$ has been separated from those for $a^{(i)}$ and $b^{(i)}$ in order to remove intervals.
b) Two more constraints have been added to determine k so that intervals with too small or too large $L^{(i)}$ will be removed.

4) *Reasonable-interval processing*: A close study of the derivation of the reasonable interval processing in [17] revealed that more results could be obtained from it; results that not only ensure overlapping intervals but also ensure that those intervals are not overly long.[5] Fig. 4 (which is adapted from [17, Fig. 19a]) depicts the situation. In [17], a threshold ξ^* was determined from probability theory, and only intervals for which $a^{(i)} < \xi^*$ and $b^{(i)} > \xi^*$ were retained. A close examination of the derivation of ξ^* reveals that [17, eqs. (A5)], whose solution is ξ^*, can be interpreted geometrically as "ξ^* *occurs at the intersection of the two normal distributions* $p(a^{(i)})$ *and* $p(b^{(i)})$." Observe that this intersection occurs when $p(a^{(i)}) = p(b^{(i)}) = t$. Observe also that this simple equation has three solutions, and not just the one at ξ^*. The two other solutions occur at

$$\begin{cases} a^{(i)} = a^* = m_a - (\xi^* - m_a) = 2m_a - \xi^* \\ b^{(i)} = b^* = m_b - (\xi^* - m_b) = 2m_b - \xi^* \end{cases} \tag{25}$$

where m_a and m_b are the mean values of the left and right endpoints of the surviving intervals. In EIA, only the intervals $[a^{(i)}, b^{(i)}]$ are kept such that

$$2m_a - \xi^* \leqslant a^{(i)} < \xi^* < b^{(i)} \leqslant 2m_b - \xi^* \tag{26}$$

where ξ^* is again computed by (1).

[5] If overly long intervals overlap by a small amount, then this is a poor indication that "words must mean similar things to different people." Our new tests are about keeping the overlapping intervals short.

TABLE III
SUMMARY OF THE DIFFERENCES BETWEEN THE EIA AND THE IA

Step	IA	EIA
Bad data processing	$0 \leqslant a^{(i)} < b^{(i)} \leqslant 10$	Extra requirement $b^{(i)} - a^{(i)} < 10$
Outlier processing	Box and Whisker tests on $a^{(i)}$, $b^{(i)}$ and $L^{(i)}$ simultaneously	Box and Whisker tests on $a^{(i)}$ and $b^{(i)}$ first and then on $L^{(i)}$
Tolerance limit processing	Tests on $a^{(i)}$, $b^{(i)}$ and $L^{(i)}$ simultaneously	Tests on $a^{(i)}$ and $b^{(i)}$ first and then on $L^{(i)}$; Two more constraints to remove too large or too small $L^{(i)}$
Reasonable-interval processing	$a^{(i)} < \xi^* < b^{(i)}$	$2m_a - \xi^* \leqslant a^{(i)} < \xi^* < b^{(i)} \leqslant 2m_b - \xi^*$
FS part	In EIA the method for computing the interior FOUs is improved	

Compared with reasonable-interval processing in IA, constraints have now been added on the lower limit of $a^{(i)}$ and the upper limit of $b^{(i)}$, both of which help us to control the breadth of the surviving intervals, as desired.

The FS part in the EIA is identical to that in the IA, except that in the final step, the procedure for computing the LMF of interior FOUs is modified to handle all cases in Fig. 9 correctly. After classifying the FOU into one of the three shapes and deleting inadmissible T1 FSs, the m interval endpoints are reduced to m^* interval endpoints, which are then used in computing the FOU parameters. The procedures for shoulder FOUs and for the UMF of interior FOUs in the IA are correct; however, the procedure for the LMF of interior FOUs needs improvement. Currently, it only considers the case that the LMF of an interior FOU is completely determined by the two embedded T1 FSs that also determine the UMF, and these two embedded T1 FSs must form the perfect triangle $[\overline{a}^{(i)}, p, \underline{b}^{(i)}]$, as shown in Fig. 5(b); however, as explained earlier, this is not always true in practice, e.g., a counterexample is shown in Fig. 9(a). The key point is to determine the location and height of the apex, i.e., p and μ_p. As discussed in Section II-C, (p, μ_p) should be the lowest intersection of the left legs of existing embedded T1 FSs with the right legs. Frequently, one has fewer than 200 such embedded T1 FSs and the EIA is always used offline; hence, one can use exhaustive search to find this apex, i.e., find all possible intersections of left legs with right legs and then choose the apex as the intersection with the minimum height in $[\overline{a}_{\rm MF}, \underline{b}_{\rm MF}]$.

A summary of the differences between the EIA and the IA is given in Table III. Observe that the data part of the EIA has more strict and reasonable tests than the IA, and the FS part of the EIA has an improved procedure to compute the LMF, more specifically, the apex of the LMF. The MATLAB code for EIA and the data from the survey are available on the authors' websites http://www-scf.usc.edu/~dongruiw/files/EIA.zip and http://sipi.usc.edu/~mendel/.

B. Examples

When the EIA was applied to the dataset introduced in Section II-B, the remaining numbers of data intervals after each processing stage are shown in Table II. Comparing m^* in the IA with m^* in the EIA, observe that the latter is smaller for each word, which is intuitive, since more strict tests are implemented in the EIA.

The resulting IT2 FSs are shown in Fig. 10, where the m^* embedded T1 FSs for each IT2 FS are also shown. The number in the title of each subfigure is the area of corresponding FOU.

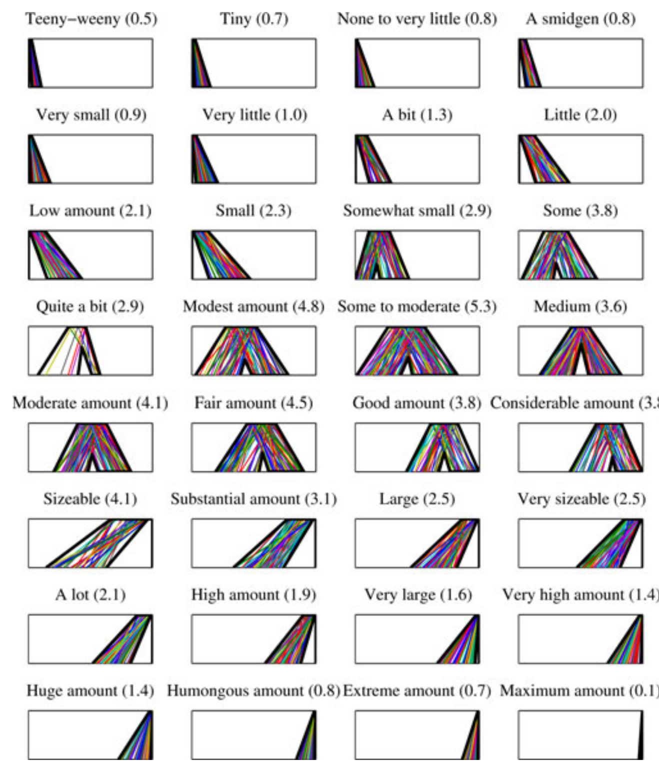


Fig. 10. IT2 FS word models obtained from the web dataset using the EIA.

Comparing Fig. 10 with Fig. 8, observe that, generally, the FOUs from EIA become thinner, and the LMFs from the EIA are higher than the corresponding quantities from the IA. Therefore, our enhancements to the IA are effective. Also observe that *Considerable amount* is a right shoulder in Fig. 8, whereas it changes to an interior FOU in Fig. 10. Additionally, *Quite a bit* in Fig. 10 has very few embedded T1 FSs. These two words are examined more closely in the next section.

To show that the difference between the EIA and the IA is statistically significant, we performed two paired t-tests [35], [56]. The first paired t-test was on the final number of data intervals that are used in the FS part, i.e., m^* in Table II. When $\alpha = 0.05$, we have $t(31) = 9.45$ and $p < 0.0001$, i.e., the difference between m^* in the EIA and the IA is statistically significant. The second paired t-test was on the area of the FOUs obtained from the two approaches, shown in the titles of the subfigures in Figs. 8 and 10. When $\alpha = 0.05$, we have $t(31) = 9.04$ and $p < 0.0001$, i.e., the difference between the area of the FOUs from the two approaches is again statistically significant.

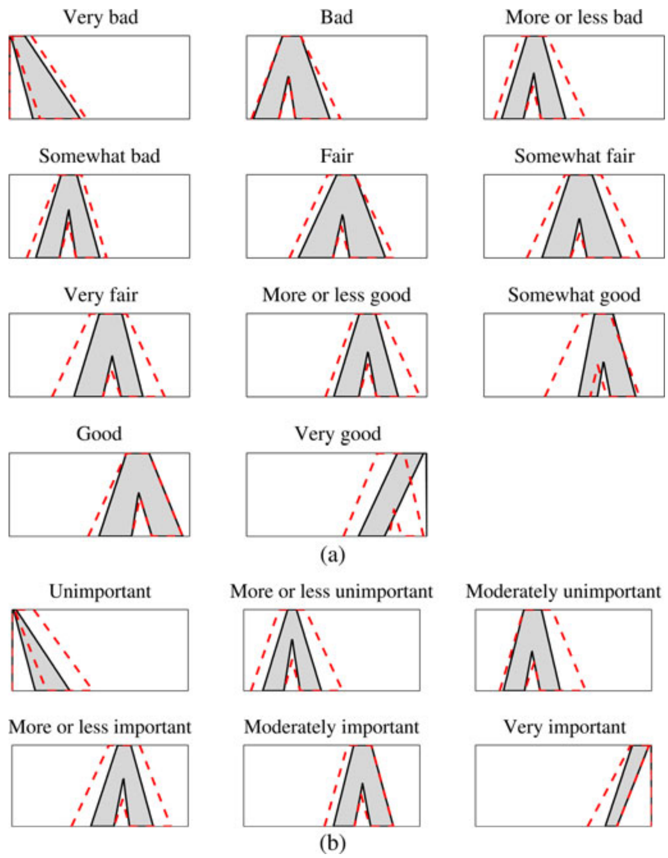


Fig. 11. IT2 FS word models for two vocabularies in the investment judgment advisor [29]. The FOUs are obtained from the EIA and the dashed red curves represent the UMFs and LMFs obtained from the IA.

We also applied both the IA and the EIA to two smaller datasets that we collected from 40 adults in 2008 for the investment judgment advisor [29]. The results are shown in Fig. 11. Clearly, the EIA resulted in thinner and narrower FOUs. Observe also that the EIA gave a right-shoulder FOU for the word "Very good" in Fig. 11(a), which seems more reasonable than the interior FOU given by the IA.

C. Discussions

We have observed that the EIA can result in thinner and narrower FOUs than the IA, and we believe the EIA FOUs are more reasonable because of the following.

1) Thinner and narrower FOUs may represent better compromise between uncertainty and accuracy. We want to use the FOUs to capture enough uncertainties, and generally, a larger FOU can capture more uncertainties. A fully filled-in granule (whose upper membership grade is 1 and lower membership grade is 0 in the entire interval [0, 10]) captures the most uncertainty but is not useful. On the other hand, we do not want to sacrifice accuracy, in which case, generally, a smaller FOU means more accuracy. A T1 FS is the most accurate IT2 FS, but it cannot capture the interpersonal uncertainty at all. The FOUs that are generated by the IA seem to contain too many uncertainties, and hence, the accuracy is poor. For example, in Fig. 8, many words (e.g., *some, modest amount, some to moderate, moderate amount, fair amount, considerable amount, sizeable, and substantial amount*) almost cover the entire [0, 10] input domain, which does not sound correct. Therefore, we would like to reduce the uncertainties that are captured by the FOUs and increase their accuracy, which result in thinner and narrower FOUs, as those shown in Fig. 10.
2) Thinner and narrower FOUs can be used to better distinguish among close words. For example, in Fig. 8, the IA FOUs for *a bit, little, low amount,* and *small* are almost identical. It is true that these words are close; however, they are not as close as their FOUs suggest, which indicates some problems with the IA. The EIA FOUs for these four words are given in Fig. 10. Observe that their FOUs are more dissimilar than those in Fig. 8. As a result, these four words can be better distinguished.

IV. Convergence Analysis of Enhanced Interval Approach

There is much randomness in the data collection and processing steps of EIA. Therefore, a natural questions follows: Can one obtain a stable FOU after collecting enough data intervals? In other words, assume there is a reference FOU model for the word that is obtained when infinitely many subjects are surveyed, and then, does the output of EIA converge to that reference model? If yes, then how many data intervals does one need to collect before the output of EIA is sufficiently similar to the reference model? This section aims to answer these two important questions.

A. Stochastic Convergence

Four popular forms of stochastic convergence are [32] *convergence in distribution*, *convergence in probability*, *convergence with probability 1*, and *convergence in mean square*. It is well known [32] that convergence in mean square implies convergence in probability (the converse is not true), and convergence in probability implies convergence in distribution (the converse is not true). In this paper, our focus is on convergence in mean square of the FOU word models. This is accomplished by testing for convergence of the similarity of the FOUs (more on this in latter part of this section).

The EIA maps the assumed random interval endpoints into an FOU. Even though this mapping is linear (see Table I), by the time the set of m^* T1 FSs is upper and lower bounded, the resulting UMF and LMF for the FOU are very nonlinear functions of the surviving m^* data intervals. This means that it is not possible to compute the mathematical probability distributions for the parameters of the FOU (and their associated population means and variances) or for the FOU (it depends jointly on all of its parameters). Instead, the FOU is viewed herein as a generic nonlinear function h of the m^* data intervals.

Another well-known fact from probability theory is the following [32]: If several random variables converge in probability to their respective true values, then a continuous function of them also converges in probability to its true value. Unfortunately, we do not know what the "true" values are for the

TABLE IV
MEAN OF $S_J(n, 175)$

Word	Number of responses, n																
	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170
Teeny-weeny	0.48	0.63	0.69	0.75	0.79	0.84	0.84	0.86	0.88	0.9	0.91	0.92	0.93	0.95	0.96	0.98	1
Tiny	0.63	0.71	0.76	0.78	0.83	0.84	0.87	0.88	0.89	0.9	0.9	0.92	0.93	0.94	0.96	0.96	0.99
None to very little	0.6	0.71	0.79	0.84	0.87	0.88	0.91	0.92	0.93	0.95	0.95	0.95	0.96	0.97	0.97	0.98	0.99
A smidgen	0.52	0.67	0.75	0.78	0.8	0.83	0.86	0.87	0.9	0.9	0.91	0.93	0.94	0.95	0.97	0.98	1
Very small	0.62	0.74	0.78	0.81	0.83	0.85	0.87	0.89	0.9	0.91	0.92	0.93	0.94	0.95	0.95	0.96	0.97
Very little	0.66	0.74	0.79	0.82	0.83	0.85	0.87	0.88	0.9	0.91	0.92	0.93	0.94	0.95	0.96	0.96	0.97
A bit	0.63	0.75	0.85	0.88	0.89	0.91	0.93	0.93	0.94	0.95	0.96	0.96	0.97	0.98	0.99	0.99	1
Little	0.69	0.81	0.86	0.89	0.91	0.92	0.93	0.94	0.95	0.96	0.96	0.97	0.97	0.98	0.98	0.98	0.99
Low amount	0.72	0.81	0.86	0.88	0.9	0.91	0.92	0.93	0.94	0.95	0.95	0.95	0.96	0.97	0.98	0.98	0.99
Small	0.73	0.82	0.86	0.87	0.9	0.91	0.92	0.93	0.94	0.95	0.95	0.96	0.97	0.97	0.98	0.99	0.99
Somewhat small	0.46	0.59	0.62	0.69	0.7	0.74	0.79	0.81	0.86	0.89	0.9	0.92	0.93	0.95	0.96	0.98	1
Some	0.43	0.58	0.69	0.75	0.78	0.82	0.83	0.85	0.85	0.86	0.87	0.88	0.88	0.89	0.91	0.94	0.98
Quite a bit	0.09	0.19	0.22	0.29	0.41	0.41	0.5	0.54	0.63	0.66	0.69	0.79	0.84	0.88	0.92	0.94	0.98
Modest amount	0.47	0.63	0.7	0.75	0.77	0.79	0.81	0.84	0.85	0.85	0.84	0.86	0.88	0.88	0.91	0.92	0.95
Some to moderate	0.64	0.77	0.83	0.85	0.87	0.88	0.91	0.92	0.91	0.94	0.94	0.95	0.97	0.98	0.98	0.99	1
Medium	0.69	0.8	0.83	0.85	0.87	0.89	0.9	0.91	0.92	0.93	0.95	0.96	0.96	0.97	0.98	0.98	0.98
Moderate amount	0.61	0.76	0.79	0.83	0.85	0.88	0.88	0.9	0.92	0.93	0.94	0.95	0.96	0.96	0.97	0.98	0.98
Fair amount	0.58	0.74	0.8	0.84	0.85	0.85	0.86	0.87	0.88	0.88	0.88	0.89	0.89	0.91	0.91	0.94	0.97
Good amount	0.45	0.56	0.61	0.64	0.66	0.69	0.72	0.77	0.8	0.83	0.84	0.88	0.91	0.94	0.95	0.96	0.97
Considerable amount	0.42	0.51	0.54	0.57	0.57	0.62	0.62	0.65	0.66	0.67	0.7	0.72	0.75	0.81	0.82	0.87	0.95
Sizeable	0.49	0.69	0.76	0.79	0.8	0.8	0.82	0.83	0.85	0.86	0.86	0.84	0.87	0.88	0.89	0.9	0.98
Substantial amount	0.66	0.76	0.82	0.85	0.88	0.9	0.92	0.93	0.94	0.94	0.95	0.96	0.96	0.97	0.98	0.99	0.99
Large	0.75	0.84	0.87	0.88	0.9	0.91	0.92	0.94	0.94	0.95	0.95	0.96	0.97	0.97	0.98	0.98	0.98
Very sizeable	0.59	0.73	0.8	0.84	0.86	0.88	0.89	0.91	0.93	0.94	0.94	0.95	0.96	0.97	0.98	0.98	0.99
A lot	0.69	0.83	0.86	0.88	0.9	0.9	0.91	0.92	0.93	0.93	0.94	0.94	0.94	0.95	0.95	0.96	0.98
High amount	0.69	0.8	0.85	0.88	0.9	0.92	0.93	0.94	0.95	0.96	0.97	0.97	0.98	0.99	0.99	0.99	1
Very large	0.6	0.74	0.79	0.83	0.85	0.86	0.87	0.89	0.89	0.9	0.91	0.91	0.92	0.93	0.94	0.95	0.96
Very high amount	0.61	0.77	0.8	0.85	0.86	0.89	0.91	0.93	0.94	0.95	0.95	0.96	0.97	0.97	0.99	0.99	1
Huge amount	0.6	0.75	0.81	0.84	0.86	0.88	0.91	0.92	0.94	0.95	0.95	0.96	0.97	0.97	0.98	0.99	0.99
Humongous amount	0.46	0.62	0.74	0.82	0.84	0.86	0.87	0.88	0.9	0.91	0.92	0.93	0.95	0.96	0.96	0.97	0.99
Extreme amount	0.63	0.74	0.82	0.85	0.87	0.89	0.91	0.91	0.92	0.92	0.93	0.93	0.94	0.94	0.94	0.95	0.98
Maximum amount	0.59	0.71	0.76	0.8	0.81	0.85	0.86	0.87	0.86	0.87	0.88	0.88	0.9	0.9	0.9	0.91	0.92
Mean $S_J(n, 175)$ of all 32 words, m_{sim}	0.58	0.7	0.76	0.8	0.82	0.84	0.86	0.87	0.89	0.9	0.91	0.92	0.93	0.94	0.95	0.96	0.98
$m_{\text{sim}} \geqslant 0.7$	3	22	25	28	28	29	30	30	30	30	30	32	32	32	32	32	32
$m_{\text{sim}} \geqslant 0.8$	0	6	14	22	25	27	28	29	29	30	30	30	31	32	32	32	32
$m_{\text{sim}} \geqslant 0.9$	0	0	0	0	3	8	13	16	16	19	24	24	26	27	30	31	32

parameters of the FOU or for the function h; hence, this result is not used by us at this time. Instead, our approach is to study the mean-square convergence of the entire FOU by using similarity numbers, as explained next. In the rest of this section, our use of the word "convergence" is synonymous with "mean-square convergence."

B. Experiment Design

A similarity measure of IT2 FSs is needed to study the convergence of the EIA. In this paper, we use the Jaccard similarity for IT2 FSs [39] shown in (27) because [39] has shown that it is the recommended similarity measure for CWW; it also has the desirable property that $s_J(\tilde{A}, \tilde{B}) = 1$

$$s_J(\tilde{A}, \tilde{B}) = \frac{\sum_{i=1}^{N} \min(\bar{\mu}_{\tilde{A}}(x_i), \bar{\mu}_{\tilde{B}}(x_i)) + \sum_{i=1}^{N} \min(\underline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{B}}(x_i))}{\sum_{i=1}^{N} \max(\bar{\mu}_{\tilde{A}}(x_i), \bar{\mu}_{\tilde{B}}(x_i)) + \sum_{i=1}^{N} \max(\underline{\mu}_{\tilde{A}}(x_i), \underline{\mu}_{\tilde{B}}(x_i))} \tag{27}$$

if and only if $\tilde{A}$ and $\tilde{B}$ are exactly the same. Therefore, to show that the output of EIA, i.e., $\tilde{A}$, converges to the reference FOU, i.e., $\tilde{B}$, we only need to show that $s_J(\tilde{A}, \tilde{B})$ converges to 1 as increasingly more data intervals are collected.

Although, theoretically, there exists a reference FOU for each word, which is obtained when infinitely many subjects are surveyed, we do not know its parameters. In this paper, we use the FOU obtained from all 175 responses as the reference and use the following procedure to study whether the FOU obtained from EIA converges to it as increasingly more data intervals are collected.

1) Randomly select (without replacement) ten responses from the 175 responses, and compute the corresponding FOU, i.e., FOU_{10}^{1}.
2) Randomly select (without replacement) another ten responses from the remaining 165 responses (excluding the ten responses in Step 1), and combine them with the previous ten responses.[6] Compute the corresponding FOU, i.e., FOU_{20}^{1}.

[6]This process is analogous to what one would do in practice. Because every survey carries a cost, in practice, one would like to construct the IT2 FS word model using the minimum number of surveys. One would first survey ten people to get ten responses and compute an FOU from it. If that FOU does not look reasonable (e.g., the FOU looks too wide, or too narrow, or too fat, or an interior (shoulder) FOU is expected, whereas a shoulder (interior) FOU is obtained), one would then survey another ten people, add their responses to the previous ten responses, and compute the FOU again. One would do this again and again until a satisfactory FOU is obtained. In this process, all responses that are obtained from previous surveys are included in the computations.

TABLE V
STD OF $S_J(n, 175)$

Word	Number of responses, n																
	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170
Teeny-weeny	0.3	0.22	0.2	0.19	0.15	0.09	0.08	0.08	0.07	0.06	0.06	0.05	0.05	0.04	0.03	0.03	0.01
Tiny	0.17	0.15	0.11	0.11	0.09	0.08	0.06	0.06	0.06	0.05	0.06	0.05	0.05	0.05	0.04	0.03	0.03
None to very little	0.24	0.22	0.16	0.15	0.12	0.1	0.07	0.07	0.06	0.04	0.04	0.04	0.04	0.04	0.03	0.02	0.02
A smidgen	0.29	0.21	0.16	0.14	0.11	0.09	0.08	0.08	0.06	0.07	0.06	0.06	0.05	0.05	0.04	0.03	0.01
Very small	0.17	0.11	0.09	0.08	0.08	0.08	0.07	0.06	0.06	0.06	0.05	0.04	0.04	0.03	0.02	0.02	0.01
Very little	0.17	0.13	0.1	0.1	0.08	0.07	0.08	0.06	0.05	0.05	0.04	0.04	0.04	0.03	0.02	0.02	0.02
A bit	0.25	0.2	0.11	0.09	0.09	0.05	0.05	0.04	0.03	0.04	0.03	0.03	0.02	0.02	0.02	0.01	0.01
Little	0.19	0.1	0.08	0.06	0.05	0.05	0.04	0.04	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.01	0.01
Low amount	0.17	0.07	0.07	0.06	0.06	0.05	0.04	0.04	0.04	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.02
Small	0.17	0.09	0.05	0.05	0.05	0.04	0.04	0.03	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.01	0.01
Somewhat small	0.14	0.12	0.13	0.15	0.15	0.15	0.15	0.14	0.13	0.11	0.1	0.09	0.08	0.07	0.04	0.03	0.01
Some	0.15	0.15	0.14	0.12	0.12	0.09	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.06	0.03
Quite a bit	0.17	0.21	0.22	0.26	0.28	0.31	0.28	0.28	0.25	0.28	0.27	0.22	0.19	0.15	0.12	0.08	0.04
Modest amount	0.16	0.14	0.12	0.1	0.1	0.09	0.09	0.07	0.07	0.07	0.07	0.07	0.06	0.07	0.06	0.05	0.05
Some to moderate	0.13	0.09	0.07	0.07	0.07	0.07	0.06	0.06	0.07	0.05	0.05	0.04	0.03	0.03	0.02	0.02	0
Medium	0.11	0.08	0.07	0.07	0.06	0.06	0.04	0.04	0.04	0.04	0.03	0.03	0.02	0.02	0.02	0.02	0.02
Moderate amount	0.15	0.09	0.08	0.08	0.07	0.06	0.06	0.06	0.05	0.05	0.04	0.03	0.03	0.03	0.02	0.02	0.01
Fair amount	0.2	0.11	0.08	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.06	0.07	0.07	0.06	0.05
Good amount	0.16	0.13	0.15	0.15	0.17	0.18	0.19	0.19	0.18	0.17	0.17	0.15	0.12	0.08	0.05	0.03	0.03
Considerable amount	0.15	0.1	0.12	0.13	0.11	0.15	0.15	0.17	0.17	0.18	0.19	0.2	0.2	0.2	0.2	0.19	0.13
Sizeable	0.29	0.22	0.18	0.18	0.18	0.19	0.18	0.17	0.15	0.16	0.16	0.17	0.17	0.17	0.16	0.15	0.05
Substantial amount	0.21	0.17	0.13	0.13	0.1	0.05	0.06	0.06	0.06	0.06	0.03	0.03	0.03	0.03	0.02	0.01	0.01
Large	0.15	0.06	0.07	0.06	0.06	0.05	0.05	0.05	0.04	0.04	0.03	0.03	0.03	0.02	0.02	0.01	0.01
Very sizeable	0.29	0.18	0.11	0.1	0.09	0.08	0.08	0.06	0.05	0.04	0.04	0.03	0.03	0.03	0.02	0.01	0.01
A lot	0.23	0.09	0.07	0.06	0.06	0.06	0.05	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.04	0.03
High amount	0.18	0.1	0.09	0.07	0.06	0.06	0.05	0.05	0.04	0.04	0.03	0.03	0.02	0.02	0.02	0.01	0
Very large	0.21	0.14	0.08	0.08	0.07	0.06	0.06	0.06	0.06	0.05	0.05	0.05	0.05	0.05	0.04	0.04	0.03
Very high amount	0.17	0.15	0.12	0.11	0.11	0.1	0.08	0.08	0.07	0.07	0.06	0.05	0.04	0.04	0.03	0.02	0
Huge amount	0.21	0.14	0.14	0.13	0.11	0.11	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.04	0.02	0.02	0.01
Humongous amount	0.34	0.31	0.24	0.16	0.16	0.14	0.14	0.09	0.08	0.07	0.06	0.05	0.04	0.04	0.03	0.03	0.02
Extreme amount	0.24	0.17	0.14	0.12	0.11	0.09	0.08	0.08	0.08	0.07	0.06	0.05	0.05	0.05	0.05	0.05	0.03
Maximum amount	0.21	0.18	0.19	0.18	0.17	0.14	0.13	0.13	0.13	0.12	0.11	0.1	0.1	0.09	0.09	0.08	0.08
Mean std of all 32 words, σ_{sim}	0.2	0.14	0.12	0.11	0.1	0.1	0.09	0.08	0.08	0.07	0.07	0.06	0.06	0.05	0.05	0.04	0.02
$\sigma_{\text{sim}} \leqslant 0.1$	0	8	13	14	18	22	25	26	26	26	26	27	28	29	29	30	31
$\sigma_{\text{sim}} \leqslant 0.05$	0	0	0	1	2	3	5	7	9	12	17	18	20	22	23	25	30
$\sigma_{\text{sim}} \leqslant 0.02$	0	0	0	0	0	0	0	0	0	0	0	0	1	2	8	13	19

3) Randomly select (without replacement) another ten responses from the remaining 155 responses (excluding the 20 responses in Step 2), and combine them with the previous 20 responses. Compute the corresponding FOU, i.e., FOU_{30}^1.
4) Repeat Steps 1–3 until FOU_{175}^1 is computed. So far, a group of nested responses $\{10, 20, \ldots, 170, 175\}$ and the corresponding $\{\text{FOU}_{10}^1, \text{FOU}_{20}^1, \ldots, \text{FOU}_{170}^1, \text{FOU}_{175}^1\}$ have been constructed.
5) Repeat Steps 1–4 to construct another 99 groups of such nested responses, and compute the corresponding $\{\text{FOU}_{10}^i, \text{FOU}_{20}^i, \ldots, \text{FOU}_{170}^i, \text{FOU}_{175}^i\}$, where $i = 2, 3, \ldots, 100$.
6) Compute the following Jaccard similarity measures:
   a) $s_J(\text{FOU}_{10}^i, \text{FOU}_{175}^i)$, $i = 1, 2, \ldots, 100$. The collection of these 100 numbers is denoted as $S_J(10, 175)$.
   b) $s_J(\text{FOU}_{20}^i, \text{FOU}_{175}^i)$, $i = 1, 2, \ldots, 100$. The collection of these 100 numbers is denoted as $S_J(20, 175)$.

   $\vdots$

   c) $s_J(\text{FOU}_{170}^i, \text{FOU}_{175}^i)$, $i = 1, 2, \ldots, 100$. The collection of these 100 numbers is denoted as $S_J(170, 175)$.

Note the two fundamental differences in experiment design between this paper and [6].

1) In [6], we sampled from the m^* (the number of data intervals after FOU classification) data intervals, whereas in this paper, we sample from the original n data intervals.
2) In [6], we sampled m_1 data intervals from m^*, and then another m_2 data intervals from m^*, where the m_1 data intervals, and the m_2 data intervals were completely independent. In this paper, the m_1 data intervals must be a subset of the m_2 data intervals when $m_1 < m_2$.

The first modification enables us to answer the question "how many data intervals are needed before the output of EIA is sufficiently similar to the reference model?," and the second modification makes our experiment design resemble the practice explained in Footnote 6. Because the present approach is always based on the original data intervals and not on a preprocessed subset of them, we believe it is more meaningful than the approach in [6].

C. Experimental Results

The mean and std of $S_J(n, 175)$, $n = 10, 20, \ldots, 170$ are shown in the first part of Tables IV and V, respectively. The second lower part of Table IV shows the mean $S_J(n, 175)$ for all 32 words (m_{sim}), i.e., the mean of each column in the first

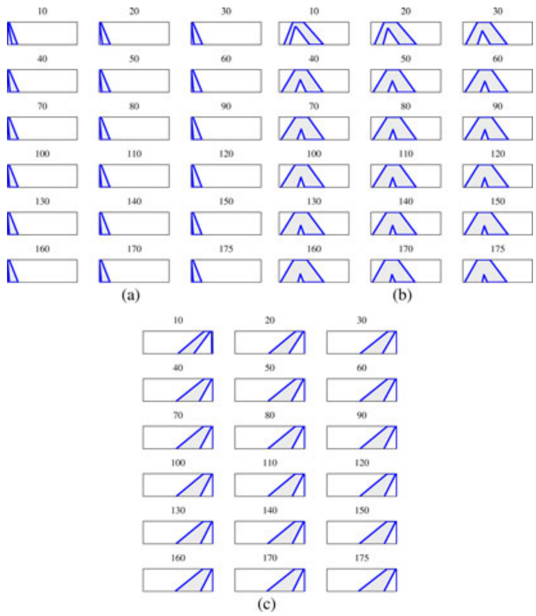


Fig. 12. Average IT2 FS models for (a) *Tiny*, (b) *Some*, and (c) *Large* when n of 175 responses are used in the EIA. n is shown at the top of each figure.

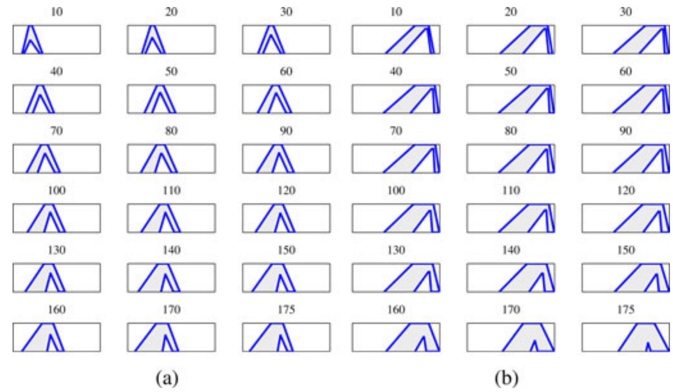


Fig. 13. Average IT2 FS models for (a) *Quite a bit* and (b) *Considerable amount* when n of 175 responses are used in the EIA. n is shown at the top of each figure.

part of the table. The third lower part of Table IV shows the number of words whose mean $S_J(n, 175)$ is larger than 0.7, 0.8, and 0.9, respectively. The second lower part of Table V shows the mean std of $S_J(n, 175)$ for all 32 words (σ_{sim}), i.e., the mean of each column in the first part of the table. The third lower part of Table V shows the number of words whose std of $S_J(n, 175)$ is smaller than 0.1, 0.05, and 0.02, respectively. We observe the following.

1) Generally, the mean of $S_J(n, 175)$ increases monotonically toward 1 when n increases.
2) Generally, the std of $S_J(n, 175)$ decreases monotonically toward 0 when n increases.
3) *Quite a bit* and *Considerable amount* seem to be the most difficult words to model, as they need more than 100 responses to obtain large $S_J(n, 175)$.

In summary, generally, the IT2 FS word models that are obtained from the EIA converge in mean-square sense as increasingly more responses are collected, and on average, 30 responses can bring the mean $S_J(30, 175)$ to 0.76 and the std of $S_J(30, 175)$ to 0.12. When $n \geqslant 30$, adding more responses only changes the mean and std of $S_J(n, 175)$ very slowly. Therefore, we suggest that 30 responses should be collected, in practice, as a good compromise between cost and accuracy.

Three words, i.e., *Tiny*, *Some*, and *Large*, are used as examples to illustrate how the FOUs look when only n ($n = 10, 20, \ldots, 170$) of the 175 responses are used in the EIA. The results are shown in Fig. 12, where each FOU is an average of the 100 realizations in the previous section, e.g., the FOU for ten responses is the average of FOU_{10}^i, $i = 1, 2, \ldots, 100$. Clearly, the FOUs visually look stable after 30 responses.

It is also interesting to observe how the FOUs of *Quite a bit* and *Considerable amount* evolve as n increases from 10 to 175. The results are shown in Fig. 13. Observe that their FOUs continue to change shape as n increases. Since different people have different understandings about *Quite a bit*, very few embedded T1 FSs survive in the EIA, as can be seen in Fig. 10. An explanation for this for *Quite a bit* is that it is composed of two opposite sounding words: *quite*, which sounds large, and *a bit*, which sounds small. The problem with *Considerable amount* is that it is at the boundary of medium-sounding and large-sounding words; therefore, it is difficult to determine whether it should be an interior FOU or a right shoulder. This suggests that the EIA may also be able to detect "linguistically difficult" words, which may be of importance in future CWW studies.

V. Conclusion

Construction of IT2 FS word models is the first step in the Per-C, which is an implementation of CWW. The IA has been, so far, the only systematic method to construct such models from data intervals that are collected from a survey; however, as has been pointed out in this paper, the IA has some limitations, and its performance can be further improved. This paper has proposed an EIA and demonstrated its performance on data collected from a web survey. The data part of the EIA has more strict and reasonable tests than the IA, and the FS part of the EIA has an improved procedure to compute the LMF, more specifically, the apex of the LMF. A convergence analysis has also been performed in order to answer two important questions: 1) Does the output IT2 FS from the EIA converge to a stable model as increasingly more data intervals are collected, and 2) how many data intervals are needed before the resulting IT2 FS word model is sufficiently similar to the model obtained from infinitely many data intervals? Our results showed that the EIA converges in a mean-square sense for most of the words, and generally, 30 data intervals seems to be a good compromise between cost and accuracy.

Since IT2 FSs are special cases of general type-2 FSs, it would be interesting to see how the EIA can be extended to generate general type-2 FS models for words. This is one of our future research directions.

APPENDIX
INTERVAL TYPE-2 FUZZY SETS

A T1 FS has membership grades that are crisp, whereas a T2 FS [21], [25]– [27], [29], [38], [51] has membership grades that are T1 FSs. Such a set is, particularly, useful in circumstances where it is difficult to determine the exact MF for an FS, e.g., approximate reasoning [7], [40], [42], recognition and classification [18], [19], [48], [55], system modeling and control [2], [3], [5], [8], [9], [12], [15], [16], [21], [36], [44]–[47], word modeling [17], [28], [29], [41], etc.

The MF of a T2 FS[7] is 3-D, with x-axis called [21] the *primary variable*, y-axis called the *secondary variable* (or *primary membership*) and z-axis called the *MF value* (or *secondary MF value*). A *vertical slice* is a plane that is parallel to the MF-value axis. A T2 FS $\tilde{A}$ is [1] a bivariate function on the Cartesian product, $X \times [0,1]$ into $[0,1]$, i.e., $\mu : X \times [0,1] \rightarrow [0,1]$, where X is the universe for the primary variable (x) of $\tilde{A}$. The 3-D MF of $\tilde{A}$ is usually denoted $\mu_{\tilde{A}}(x,t)$, where $x \in X$ and $t \in [0,1]$. The 2-D support of μ is called the FOU of $\tilde{A}$, i.e., [1]

$$\text{FOU}(\tilde{A}) = \{(x,t) \in X \times [0,1] | \mu_{\tilde{A}}(x,t) > 0\} \quad (28)$$

where FOU($\tilde{A}$) is bounded by lower and upper bounding functions (MFs), which are denoted $\underline{\mu}_{\tilde{A}}(x)$ and $\overline{\mu}_{\tilde{A}}(x)$, respectively, where [1]

$$\underline{\mu}_{\tilde{A}} = \inf\{t | t \in [0,1], \mu_{\tilde{A}}(x,t) > 0\} \quad (29)$$

and

$$\overline{\mu}_{\tilde{A}} = \sup\{t | t \in [0,1], \mu_{\tilde{A}}(x,t) > 0\}. \quad (30)$$

The primary membership of $\tilde{A}$, which is denoted J_x, is the interval $[\underline{\mu}_{\tilde{A}}(x), \overline{\mu}_{\tilde{A}}(x)]$, i.e., [1]

$$J_x = \{t \in [0,1] | \mu_{\tilde{A}}(x,t) > 0\} \quad (31)$$

where $\mu_{\tilde{A}}(x,t)$ is called the secondary grade of $\tilde{A}$. The secondary MF of $\tilde{A}$ is denoted[8] $\mu_{\tilde{A}}(x)$, $\mu_{\tilde{A}}(x|t)$, or $\int_{x \in J_x} \mu_{\tilde{A}}(x,t)/x$, and is [1] the restriction of function $\mu_{\tilde{A}}(x,t) : X \times [0,1] \rightarrow [0,1]$ to $x \in X$; it is also called a vertical slice of $\mu_{\tilde{A}}(x,t)$.

An embedded T1 FS, i.e., A_e, is a function whose range is a subset of $[0,1]$ determined by $\mu_{\tilde{A}}(x,t)$, i.e., [1]:

$$A_e = \{(x, t(x) | x \in X, t \in J_x)\}. \quad (32)$$

When both the primary and secondary variables are discretized, as is done during computations involving T2 FSs, there will be n_A embedded T1 FSs that are contained within FOU($\tilde{A}$). An embedded T2 FS, i.e., $\tilde{A}_e$, uses A_e as its 2-D domain and has associated secondary grades for that set, i.e.,

$$\tilde{A}_e = \mu_{\tilde{A}}(A_e). \quad (33)$$

[7]Different notations can be used to do this. [1, Tab. I] delineates the "fuzzy set notation" (which has been used in hundreds of articles) and the "standard mathematical notation" (which is more precise). In this paper, a mixture of the two notations is used, and the materials in this section are taken from [20].

[8]Analogous to many books on probability in which the explicit formula for a pdf is given only for nonzero values of its independent variable(s), and the pdf is zero for all other values of its independent variable(s), $\mu_{\tilde{A}}(x)$ is zero *by convention* for all $x \in [0,1]$ and $x \notin J_x$.

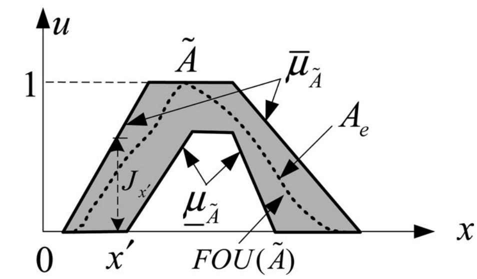


Fig. 14. IT2 FS and its associated quantities.

Mendel and John provided a Representation Theorem [26] for $\tilde{A}$: $\tilde{A}$ is the (set theory) union of all of its embedded T2 FSs. Although impractical for computation, this representation of a T2 FS has proved to be of great value to develop new theoretical results.

An IT2 FS is a T2 FS all of whose secondary grades equal 1. It is [1] a function on X into $D \in [0,1]$, where D is the set of closed subintervals of $[0,1]$, i.e., $\mu_{\tilde{A}}(x,t) : X \rightarrow D \subset [0,1]$. Because the secondary grades are all the same, they convey no useful information for the IT2 FS; hence, the IT2 FS is completely described by its FOU and, consequently, by its LMF and UMF. The Representation Theorem for an IT2 FS [27] is that its FOU is the (set theory) union of all of its embedded T1 FSs, i.e., its FOU is covered by the union of all of its embedded T1 FSs.

An example of the FOU of an IT2 FS is depicted in Fig. 14. Also shown on this figure are the LMF and UMF for such an FOU, as well as an example of an embedded T1 FS.

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