Critique of "A New Look at Type-2 Fuzzy Sets and Type-2 Fuzzy Logic Systems" [1]

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***Abstract*—This letter provides a critical review of "A New Look at Type-2 Fuzzy Sets and Type-2 Fuzzy Logic Systems" *IEEE Trans on Fuzzy Systems*, and debunks its four claims.**

***Index Terms*—Footprint of uncertainty, interval type-2 fuzzy sets, type-2 fuzzy sets, type-2 fuzzy logic systems**

[1] provides a so-called "new look" at type-2 fuzzy sets and systems and makes some very strong claims about the superiority of the new look over the present look. The purpose of this Letter is to provide a critical review of [1] that will debunk the claims made in it.

[1] is filled with new notations, definitions, concepts, theorems, many examples, and a simulation. As such, it is very easy for a reader to be seduced by all of this because of the dogmatic style of the article and the author's very important prior contributions to the type-1 field. Its Concluding Remarks section summarizes [1]'s contributions as a set of four claims that have provided us with what we believe is a good way to construct this critical review. Before doing this, however, we would like to summarize what the author's intent is. It is to provide a new representation for a type-2 fuzzy set (T2 FS) and to show how a T2 FS can be reduced (marginalized) to a type-1 (T1) FS, so that a T2 fuzzy logic system (FLS) can be reduced to a T1 FLS, after which nothing new is needed because all computations will only involve T1 FSs. This is the well-known mathematical technique of solving a problem by reducing it to one for which a solution already exists.

Claim 1 [1]: *Conditional fuzzy sets provide a mathematically more convenient and conceptually clearer framework than the type-2 fuzzy sets to study the dependence of one fuzziness on the other.*

[1] correctly attributes the following definition of a T2 FS to Zadeh [2]: *A T2 FS is a FS whose membership values are T1 FSs*. What Zadeh did not do was to explain how to accomplish this, so there may be different ways to do it.

For more than 40 years the T2 community has developed mathematical models of T2 FSs that have exemplified Zadeh's statement that *membership values are T1 FSs*. This can be explained in different ways, but here is one way that has found great utility during the past 17 years: begin with a T1 FS, X, and blur its membership function (MF), obtaining a footprint of uncertainty[1] (FOU), and then assign either a uniform or non-uniform weighting to the FOU. We will summarize this succinctly, as: *T1, blur[2] and weight*. Uniform weighting leads to an interval T2 (IT2) FS, whereas non-uniform weighting leads to a general T2 (GT2) FS. This explanation builds upon what one already knows, namely a T1 FS, and is very easy to understand, to visualize and to express mathematically. Notice that this is an explanation that is directly in the MF domain, in full agreement with Zadeh, and lets one go from a T1 FS MF to a T2 FS MF. Examples of three easy to understand FOUs are depicted in Fig. 1.

[1] introduces a different representation for a T2 FS in its Theorem 1, which in itself is an intellectual achievement. This model also begins with a T1 FS, X, but is one that is conditioned on parameters (V) that define a conditional T1 FS, $X\,|\,V$. It then assumes that these parameters are themselves fuzzy sets. Notice that this model no longer is in full agreement with Zadeh, and at this point does not let one go directly from a T1 FS to a T2 FS MF (more about this later). Additionally, assuming that MF parameters can be modeled as T1 FSs is a very seductive assumption because, except for using interval fuzzy numbers as a model for a MF parameter (which easily connects to the physically meaningful statement that a parameter varies between two numbers, which is also an interval), one would be very hard pressed to justify such an assumption, especially for T1 FSs that are described by more than one parameter, as is the case for many familiar T1 FSs (e.g., as in Fig. 1). To put this another way, [1]'s Theorem 1 is valid only if a fuzzy set model for V exists. We ask the author of [1] to provide such a model for the three FOUs in Fig. 1.

Equation (9) in [1], which is derived from Zadeh's Extension Principle, constructs the secondary MFs of a T2 FS (which are T1 FSs), but it requires solving an optimization problem. Example 3 in [1] illustrates the calculations for a Gaussian T1 MF whose center is assumed to be a T1 FS that is also described by a Gaussian MF. Although the solution to this specific optimization problem is obtained in closed-form, it requires a considerable effort; but, we are sure that any reader of this Transaction could do this.

What is totally missing from [1] are: (1) the extension of the new representation from one to more than one MF parameter, and (2) an examination of the connections of this new

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[1] For simplicity here, we are only focusing on T2 FSs whose primary and secondary variables are continuous and whose FOU is closed. See [3] and [4] for more discussions about this.

[2] Although the word "blur" is not very technical, those who work with T2 FSs have no difficulty in understanding it and are able to translate the results of blurring into mathematics. For the FOUs in Fig. 1, this is accomplished by writing formulas for the lower and upper bounding functions of each FOU, and for the secondary MFs (the MFs that provide the weights for the FOU). Secondary MFs are not shown in Fig. 1.

representation of a T2 FS to (at the very least) the popular *T1, blur and weight* representation.

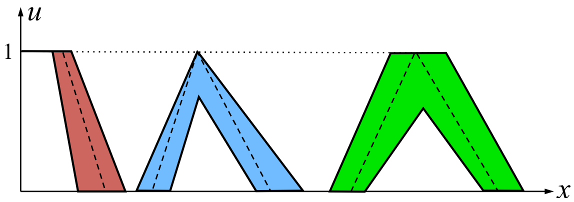


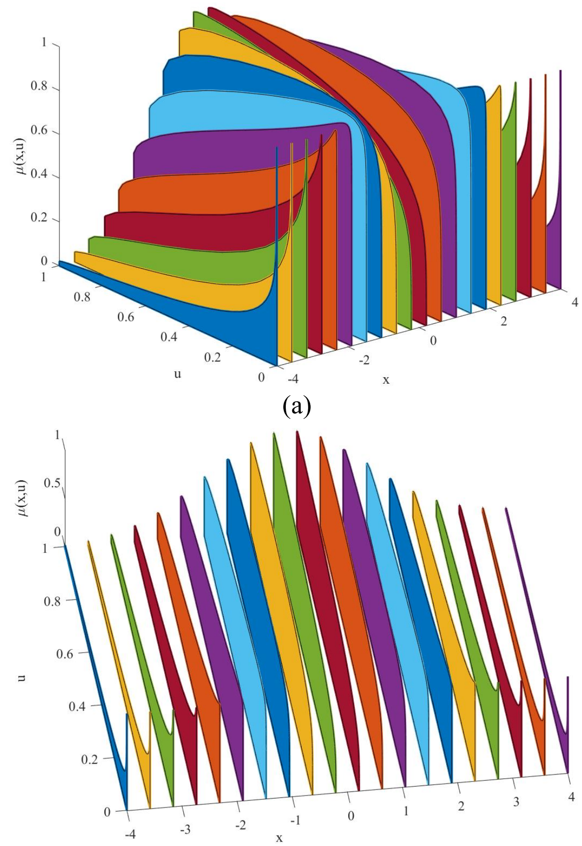
Fig. 1. Shoulder and two interior FOUs for an IT2 FS. The dashed lines in each FOU denote a candidate T1 FS that has been "blurred" to give the FOU. Symmetry of the FOU about such a T1 FS is not required.

One may argue that the extension of the new representation from one to more than one MF parameter is conceptually straightforward, but to so (and it is never even touched upon in [1]) would be disingenuous, because (9) would now require solving a separable multivariable optimization problem, and, even if this can be done, it would lead to very complicated formulas for the secondary MFs that we conjecture would provide no insight into the nature of the T2 FS and its connection back to the original T1 FS.

After more than 40 years of publications about T2 FSs it would seem very important to try and compare or relate any new representation of a T2 FS to (at the very least) the *T1, blur and weight* representation. For example, what do the MFs of [1]'s T2 FSs look like? Figure 2 illustrates what one such MF looks like; it was computed using (9) in [1], for the just mentioned Example 3 in [1] (when $c = 0$ and $\sigma_1 = \sigma_2 = 1$). From Fig. 2 one can see that the MF for each x is a T1 FS, so it is in agreement with Zadeh [1]; however, this GT2 FS is very strange, e.g., it assigns maximum uncertainty to the vertical slices in the vicinity of x = 0, all of the primary memberships cover all of [0,1], and the shapes of the secondary MFs are bizarre. We see no connection between this MF and one that would have been obtained for *T1, blur and weight* representation. This suggests to us that this new representation of a T2 FS, while mathematically correct, leads to T2 FSs that are of questionable value. It only takes one example to cast doubt on the meaningfulness of a new kind of T2 FS representation, especially when the author of that new model claims that it is <u>the</u> correct representation.

We suggest that [1] has bypassed an examination of the connections of the new representation of a T2 FS to the popular *T1, blur and weight* representation because of what [1] does next with the new T2 FS representation, namely [1] reduces it to a T1 FS; so in [1] the T2 FS is merely a means to an end, whereas to the T2 FS community the T2 FS is the starting point for everything that follows.

Before moving on to Claim 2, we would also like to point out that (14) in [1], which is the formula that gives the MF of a T2 FS point by point over its two-dimensional domain, requires the specification of a t-norm. The *T1, blur and weight* representation does not require this, nor does it require solving an optimization problem, so it is a simpler representation than the one in [1]. Additionally, (14) does not seem to be in agreement with (9), which means[3] that [1] has two formulas that can be used to construct the MFs of a T2 FS but that give different results. We ask the author to explain this.



(a)

(b)

Fig. 2. Two views of the T2 MF in Example 3 in [1].

The points we have made above demonstrate that the new T2 fuzzy set representation in [1] is neither mathematically more convenient or is a conceptually clearer framework than the widely used *T1, blur and weight* representation. So much for Claim 1.

Claim 2 [1]: *Through detailed mathematics and side-by-side comparisons, we proved that the type-2 fuzzy set framework could be replaced by the conditional fuzzy set framework without any loss of generality*.

To begin, we must alert the readers to the fact that this claim has nothing to do with side-by-side comparisons of the new T2 representation and the *T1, blur and weight* representation. Instead, after [1]'s new representation is introduced, the effects of the fuzzy parameters are removed in a step that is analogous to obtaining a marginal pdf from a joint pdf, a step that leads to a marginal T1 FS. To do this, [1] uses Zadeh's Compositional Rule of Inference, but only for the minimum t-norm. Then, with some more analysis, [1] obtains its Eq. (19), an equation that shows that there are two mathematically equivalent ways to obtain the marginal T1 FS. One way uses the conditional FS and the T1 MF parameter FS, whereas the other way uses the above-mentioned Eq. (14). So, what [1] has

[3] The authors would like to thank Dr. Nilesh Karnik for pointing out a problem with Eq. (14).

shown is that the same results can be obtained in two ways but *only for the T2 FSs that result from [1]'s new representation*, which in most cases has no connection to a T2 FS people have obtained by using the *T1, blur and weight* representation.

Next, we would like to illustrate a peculiar result. Example 7 (like Example 3) in [1][4] begins with the following Gaussian MF for the conditional FS $X|V$, $\mu_{X|V}(x|V) = \exp[-|x-V|^2/\sigma_1^2]$. The center V is a Gaussian fuzzy number with MF $\mu_V(v) = \exp[-|v-c|^2/\sigma_2^2]$. The marginal T1 FS X of $X|V$ is computed by (16) in [1], as $\mu_X(x) = \exp[-|x-c|^2/(\sigma_1+\sigma_2)^2]$. Observe that because $\sigma_1+\sigma_2$ only appears additively in $\mu_X(x)$, when it comes time to optimize the parameters of this MF (during a tuning process) when it is used in a FLS, we cannot optimize σ_1 and σ_2 individually. We can only optimize their sum, so we may as well just call that sum σ^2 . Doing this, we are right back where we started, namely a T1 FS whose center and variance both need to be determined. The generality of independent σ_1^2 and σ_2^2 has been lost!

When a sweeping claim like Claim 2 is made, it only takes one counter example to disprove it, and we have just provided such a counter-example.

Before we leave this claim, we would like to mention that one of the reasons for using a T2 FS in a T2 FLS is to compute the uncertainties about the MFs as they flow through all of the input-output computations of the T2 FLS. [1] has based the use of a conditional MF on probability in which one cancels out a random variable by a mathematical integration that makes use of a conditional pdf and a marginal pdf, to obtain a pdf that no longer depends on a random variable. In probability, if a system contains, e.g., two random variables, then it is common practice to use the joint pdf, and to first carry out all computations from the input to the output of the system using the joint pdf, and to then (if one needs to do this) compute the marginal pdf of the output pdf. It is very rare to integrate out one of the random variables before computing the pdf of the output. And yet, this is precisely what [1] claims can be done in a FLS "without any loss of generality."

As a result of the points we have made, we believe that we have demonstrated that Claim 2 is false.

Claim 3 [1]: *We explained why the correct way to construct a type-2 fuzzy logic system is to first determine the marginal fuzzy sets of the type-2 (conditional) fuzzy sets in the rules so that the type-2 fuzzy rules become the conventional type-1 fuzzy rules, and to then use the standard methods of constructing type-1 fuzzy logic systems to construct the final fuzzy logic system.*

Observe in the statement of this claim the phrase "*the correct way*" which means there is only one correct way. Even if [1] had re-phrased this more conservatively, as "*a correct way*", our example that disproved Claim 2 also serves to disprove this sweeping claim.

Actually the idea of reducing the T2 FSs in a T2 FLS to T1 FSs before performing the inference is not new. In 2005 Wu and Tan [5] proposed such an approach for IT2 FLSs. Instead of using a fixed marginal T1 FS for each IT2 FS, they used an optimization procedure to find the best *equivalent T1 FS*, which is adaptive to different inputs. Note that this *adaptiveness*—meaning that the equivalent T1 FSs that are used to compute the output change as input changes—is a very unique and important property of IT2 FLSs [6]. However, it is lost in the marginal FS approach.

Even though there does not yet seem to be a direct connection between the author's new kind of T2 FSs and T1, blur and weight T2 FSs, we suggest that this representation might lead to something novel if it is used within the framework of existing T2 FLS theory. Perhaps (although, we doubt this) it might lead to a more parsimonious representation of a GT2 FS than is obtained by using the *T1, blur and weight* representation.

Space limitations prevent us from commenting on some of [1]'s jabs at T2 FSs and FLSs that are given in its Motivations 1, 2 and 3, since we have just demonstrated that all of the earlier claims are false.

Claim 4 [1]: *The conditional fuzzy set approach in this paper can be extended to the general type-n fuzzy set problem in a straightforward manner.*

Since we have falsified (debunked) Claims 1, 2 and 3, this falsifies Claim 4 as well.

Conclusions: When Zadeh invented fuzzy sets, the Bayesian community claimed that anything that he could do with a fuzzy set they could do with (subjective) probability. Time and scholarship have proven them to be wrong. It is surprising to us that an analogous claim (or claims) has been made about T2 FSs and FLSs. We believe that we have debunked these claims in this Letter.

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[4] Example 7 is a continuation of Examples 1, 2, 3 and 4, and so these collectively may be considered the main example of [1].