Computing With Words for Hierarchical Decision Making Applied to Evaluating a Weapon System

Dongrui Wu, *Member, IEEE*, and Jerry M. Mendel, *Life Fellow, IEEE*

***Abstract*—The perceptual computer (Per-C) is an architecture that makes subjective judgments by computing with words (CWWs). This paper applies the Per-C to hierarchical decision making, which means decision making based on comparing the performance of competing alternatives, where each alternative is first evaluated based on hierarchical criteria and subcriteria, and then, these alternatives are compared to arrive at either a single winner or a subset of winners. What can make this challenging is that the inputs to the subcriteria and criteria can be numbers, intervals, type-1 fuzzy sets, or even words modeled by interval type-2 fuzzy sets. Novel weighted averages are proposed in this paper as a CWW engine in the Per-C to aggregate these diverse inputs. A missile-evaluation problem is used to illustrate it. The main advantages of our approaches are that diverse inputs can be aggregated, and uncertainties associated with these inputs can be preserved and are propagated into the final evaluation.**

***Index Terms*—Computing with words (CWWs), hierarchical decision making, interval type-2 fuzzy sets, linguistic weighted averages, missile-evaluation problem, novel weighted averages (NWAs), perceptual computing.**

I. Introduction

ZADEH coined the phrase "*computing with words*" (CWWs) [89], [90], which is [90] *"a methodology in which the objects of computation are words and propositions drawn from a natural language."* Words in the CWW paradigm may be modeled by type-1 fuzzy sets (T1 FSs) [43], [87] or their extension, interval type-2 (IT2) FSs [43], [47], [49], [88]. Therefore, an inevitable question follows: Which FS model should be used in CWW?

There are at least two types of uncertainties associated with a word [45], [67]: *intrapersonal uncertainty* and *interpersonal uncertainty*. Intrapersonal uncertainty describes [45] "*the uncertainty that a person has about the word*." It is also explicitly pointed out by Wallsten and Budescu [67] that *"except in very special cases, all representations are vague to some degree in the minds of the originators and in the minds of the receivers,"* and they suggest to model it by a T1 FS. Interpersonal uncertainty describes [45] "*the uncertainty that a group of people have about the word*." It is pointed out by Mendel [43] as *"words mean different things to different people,"* and by Wallsten and Budescu [67] as *"different individuals use diverse expressions to describe identical situations and understand the same phrases differently when hearing or reading them."* Because an IT2 FS has a footprint of uncertainty (FOU) that can be viewed as a group of T1 FSs (see Fig. 1), it can model both types of uncertainty [45]; hence, we suggest that IT2 FSs be used in CWW [43], [45]. In addition, Mendel [46] has explained why it is scientifically incorrect to model a word using a T1 FS, i.e., 1) a T1 FS for a word is well defined by its membership function (MF) that is totally certain once all of its parameters are specified; 2) words mean different things to different people, and, therefore, are uncertainties; and 3) it is a contradiction to say that something certain can model something that is uncertain.

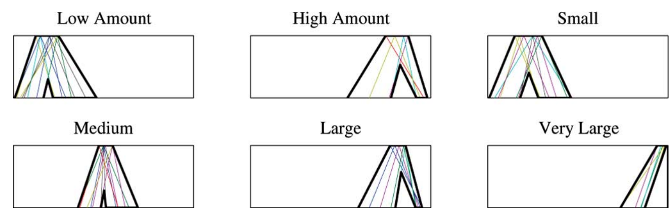

Fig. 1. Six examples of word FOUs obtained by using the IA [37] on survey results from 28 subjects. The areas between the thick curves are FOUs, and the curves within the FOUs are T1 FSs mapped from individuals' endpoint data using the IA.

CWW that use T1 FSs has been studied by many researchers, e.g., [6], [24]–[27], [32], [40], [58], [59], [63], [68], [69], [71], [84]–[86], [89], [90], and [93]; however, because of the aforementioned arguments, in this paper, IT2 FSs [4], [19], [23], [42], [43], [47]–[49], [52], [53], [56], [72], [82], [88] are used to model words.

A specific architecture, which was proposed in [44] to make subjective judgments by CWW, is shown in Fig. 2. It is called a *perceptual computer*—Per-C for short. In Fig. 2, the *encoder*[1] transforms linguistic perceptions into IT2 FSs that activate a *CWW engine*. The CWW engine performs operations on the IT2 FSs. The *decoder*[2] maps the output of the CWW engine into a recommendation, which can be a word, rank, or class.

To operate the Per-C, one needs to solve the following problems.

Manuscript received March 22, 2009; revised July 20, 2009, October 22, 2009, and January 20, 2010; accepted January 31, 2010. Date of publication February 17, 2010; date of current version May 25, 2010.

D. Wu was with the Signal and Image Processing Institute, Ming Hsieh Department of Electrical Engineering, University of Southern California, Los Angeles, CA 90089-2564 USA. He is now with the Institute for Creative Technologies and the Signal Analysis and Interpretation Laboratory, Viterbi School of Engineering, University of Southern California, Los Angeles, CA 90089-2564 USA (e-mail: dongruiw@usc.edu).

J. M. Mendel is with the Signal and Image Processing Institute, Ming Hsieh Department of Electrical Engineering, University of Southern California, Los Angeles, CA 90089-2564 USA (e-mail: mendel@sipi.usc.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TFUZZ.2010.2043439

[1]Zadeh calls this *constraint explicitation* in [89] and [90]. In [91], [92], and some of his recent talks, he calls this *precisiation*.

[2]Zadeh calls this *linguistic approximation* in [89] and [90].

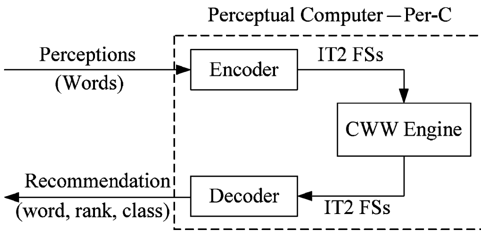

Fig. 2. Conceptual structure of the Per-C.

1) *How to transform words into IT2 FSs, i.e., the encoding problem:* This can be done with Liu and Mendel's *interval approach* (IA) [37]. First, for each word in an application-dependent encoding vocabulary, a group of subjects are asked the following question:

   On a scale of 0–10, what are the endpoints of an interval that you associate with the word ___?

   After some preprocessing, during which some intervals (e.g., outliers) are eliminated, each of the remaining intervals is classified as either an interior, left-shoulder, or right-shoulder IT2 FS. Then, each of the word's data intervals is individually mapped into its respective T1 interior, left-shoulder, or right-shoulder MF, after which, the union of all of these T1 MFs is taken. The result is an FOU for an IT2 FS model of the word. The words and their FOUs constitute a *codebook.*

   Software for the IA can be downloaded from the author's website at http://sipi.usc.edu/mendel. Therefore, the only thing that a practitioner needs to do is to conduct the survey and then to feed the data into the software. IT2 FSs are generated automatically.

   Note that in a decision-making problem, the parameters of IT2 FSs are determined as a result of the word survey. They should not be modified, because the survey results capture people's understanding about the meanings of words, and hence, they should not be distorted. In addition, to optimize the parameters of IT2 FSs, training data are needed, but they are usually not available in decision-making problems. This is different from many other IT2 fuzzy-logic systems, like fuzzy-logic modeling and control [8], [22], [23], [41], [60], [80], [81], [95], where training data are used to optimize the parameters of the IT2 FSs. For such applications, people may assign linguistic terms to the IT2 FSs to facilitate understanding; however, these linguistic terms are only used as symbols and are not necessarily the true meanings of the IT2 FSs.

2) *How to construct the CWW engine, which maps IT2 FSs into IT2 FSs:* There are different kinds of CWW engines.
   a) The *linguistic weighted average* [52], [72], [73], [75], which is defined as

$$\tilde{Y} = \frac{\sum_{i=1}^{N} \tilde{X}_i \tilde{W}_i}{\sum_{i=1}^{N} \tilde{W}_i} \tag{1}$$

   where $\tilde{X}_i$, which are the subcriteria (e.g., data, features, decisions, recommendations, judgments, scores, etc.), and $\tilde{W}_i$, which are the weights, are usually words modeled by IT2 FSs; however, they can also be special cases of IT2 FSs, e.g., numbers, intervals, or T1 FSs. It is shown in Appendix C that the upper membership function (UMF) of $\tilde{Y}$ is a fuzzy weighted average [36] of the UMFs of $\tilde{X}_i$ and $\tilde{W}_i$, and the lower membership function (LMF) of $\tilde{Y}$ is a fuzzy weighted average of the LMFs of $\tilde{X}_i$ and $\tilde{W}_i$.

   b) *Perceptual reasoning (PR)* [50], [52], [72], [79], which considers the following problem.

   Given a rule base with K rules, each of the form

$$R^k: \textit{If } x_1 \textit{ is } \tilde{F}_1^k \textit{ and } \ldots \textit{ and } x_p \textit{ is } \tilde{F}_p^k \quad \textit{Then } y \textit{ is } \tilde{G}^k \tag{2}$$

   where $\tilde{F}_j^k$ and $\tilde{G}^k$ are words modeled by IT2 FSs and a new input $\tilde{\mathbf{X}}' = (\tilde{X}_1, \ldots, \tilde{X}_p)$ with $\tilde{X}_j$ $(j = 1, \ldots, p)$ being words modeled by IT2 FSs, then what is the output IT2 FS $\tilde{Y}_{PR}$?

   In similarity-based PR [52], [72], [79], one computes

$$\tilde{Y}_{\mathrm{PR}} = \frac{\sum_{k=1}^{K} f^k(\tilde{\mathbf{X}}')\tilde{G}^k}{\sum_{k=1}^{K} f^k(\tilde{\mathbf{X}}')} \tag{3}$$

   where $f^k(\tilde{\mathbf{X}}')$ is the firing level of R^k, i.e.,

$$f^k(\tilde{\mathbf{X}}') = \prod_{j=1}^{p} s_J(\tilde{X}_j, \tilde{F}_j^k) \tag{4}$$

   in which $s_J(\tilde{X}_j, \tilde{F}_j^k)$ is the Jaccard similarity for IT2 FSs [76]

$$s_J(\tilde{X}_j, \tilde{F}_j^k) = \frac{\int_X \min(\overline{X}_j(x), \overline{F}_j^k(x))dx + \int_X \min(\underline{X}_j(x), \underline{F}_j^k(x))dx}{\int_X \max(\overline{X}_j(x), \overline{F}_j^k(x))dx + \int_X \max(\underline{X}_j(x), \underline{F}_j^k(x))dx}. \tag{5}$$

   Another approach that uses firing intervals instead of firing levels is described in [50].

3) *How to map the output of the CWW engine into a recommendation, i.e., the decoding problem:* Thus far, there are three kinds of decoders according to three forms of recommendations [52], [72].
   a) *Word:* To map an IT2 FS into a word, it must be possible to compare the *similarity* between two IT2 FSs. The Jaccard similarity measure described by (5) can be used to compute the similarities between the CWW engine output and all words in the codebook. Then, the word with the maximum similarity is chosen as the decoder's output.
   b) *Rank:* Ranking is needed when several alternatives are compared to find the best one. Because the performance of each alternative is represented by an IT2 FS obtained from the CWW engine, a ranking

TABLE I
CRITERIA WITH THEIR WEIGHTS, SUBCRITERIA WITH THEIR WEIGHTS, AND SUBCRITERIA DATA FOR THE THREE COMPANIES (ADAPTED FROM [11] AND [13])

Item	Weighting	Company A	Company B	Company C
Criterion 1: Tactics	$\tilde{9}$			
1. Effective range (km)	$\tilde{7}$	43	36	38
2. Flight height (m)	$\tilde{1}$	25	20	23
3. Flight velocity (M. No)	$\tilde{9}$	0.72	0.80	0.75
4. Reliability (%)	$\tilde{9}$	80	83	76
5. Firing accuracy (%)	$\tilde{9}$	67	70	63
6. Destruction rate (%)	$\tilde{7}$	84	88	86
7. Kill radius (m)	$\tilde{6}$	15	12	18
Criterion 2: Technology	$\tilde{3}$			
8. Missile scale (cm) (l×d–span)	$\tilde{4}$	521×35–135	381×34–105	445×35–120
9. Reaction time (min)	$\tilde{9}$	1.2	1.5	1.3
10. Fire rate (round/min)	$\tilde{9}$	0.6	0.6	0.7
11. Anti-jam (%)	$\tilde{8}$	68	75	70
12. Combat capability	$\tilde{9}$	Very Good	Good	Good
Criterion 3: Maintenance	$\tilde{1}$			
13. Operation condition requirement	$\tilde{5}$	High	Low	Low
14. Safety	$\tilde{6}$	Very Good	Good	Good
15. Defilade[a]	$\tilde{2}$	Good	Very Good	Good
16. Simplicity	$\tilde{3}$	Good	Good	Good
17. Assembly	$\tilde{3}$	Good	Good	Poor
Criterion 4: Economy	$\tilde{5}$			
18. System cost (10,000)	$\tilde{8}$	800	755	785
19. System life (years)	$\tilde{8}$	7	7	5
20. Material limitation	$\tilde{5}$	High	Low	Low
Criterion 5: Advancement	$\tilde{7}$			
21. Modularization	$\tilde{5}$	Average[b]	Good	Average[b]
22. Mobility	$\tilde{7}$	Poor	Very Good	Good
23. Standardization	$\tilde{3}$	Good	Good	Very Good

[a] *Defilade* means to surround by defensive works so as to protect the interior when in danger of being commanded by an enemy's guns.
[b] The word *general* used in [13] has been replaced by the word *average*, because it was not clear to us what *general* meant.

method for IT2 FSs is needed. A centroid-based ranking method for IT2 FSs is described in [76].

c) *Class:* A classifier is necessary when the output of the CWW engine needs to be mapped into a decision category [51]. Subsethood [52], [66], [72], [78] is useful for this purpose. The subsethood of the CWW engine output for each of the possible classes is computed first. Then, the final-decision class is the one corresponding to the maximum subsethood.

We are going to demonstrate the Per-C methodology to assist in hierarchical decision making by using a specific application, namely, a missile-evaluation problem, but this methodology is applicable to hierarchical decision-making problems in general.

There are many publications on hierarchical multicriteria decision making that use FSs [9], [17], [28], [34], [38], [62], [64], [65], [70], [83]; however, none of them have used IT2 FSs, nor have they collected data from a group of subjects to obtain the FSs. These shortcomings are overcome in our approach.

The rest of this paper is organized as follows. Section II introduces the hierarchical multicriteria missile-system-evaluation problem, Section III introduces the novel weighted averages (NWAs) that are used as the CWW engine in the Per-C, Section IV describes details about our perceptual computing approach for the missile-evaluation problem and its results, Section V compares our approach with several previous approaches, Section VI draws conclusions, and the Appendix provides algorithmic details about the NWAs.

II. MISSILE-EVALUATION PROBLEM

A tactical missile-evaluation problem is introduced in this section. We used it because it has already appeared in several publications [10]–[13], [54], and the published evaluations range from numbers to words.

A missile is a self-propelled projectile used as a weapon. It can be broadly classified into two categories [1], [18], [94]: 1) strategic missiles, which are designed for mass destruction, e.g., nuclear missiles, and 2) tactical missiles, which are designed for short-range (typically less than 300 km) battlefield use.

Tactical missiles are usually mobile to ensure survivability and quick deployment, as well as carry a variety of warheads to target enemy facilities, assembly areas, artillery, and other targets behind the front lines. Warheads can include conventional high explosive, chemical/biological, and nuclear warheads.

Tactical missiles are evaluated based on a number of different criteria [3], [12], [15], [18]. Usually, an ensemble of test missiles are fired to measure their physical properties, e.g., effective range, flight height, flight velocity, kill radius, reaction time, etc.

In this paper, a contractor has to decide which of the three companies (A, B, or C) is going to get the final mass-production contract for a tactical missile system. The contractor uses five criteria to base his/her final decision (see the first column in Table I), namely, *tactics*, *technology*, *maintenance*, *economy,* and *advancement*. Each of these criteria has some associated technical subcriteria, e.g., for *tactics,* there are seven subcriteria,

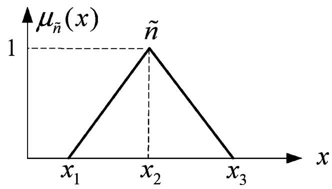

Fig. 3. MF for a fuzzy number $\tilde{n}$ (see Table II).

TABLE II
FUZZY NUMBERS AND THEIR CORRESPONDING MFS [10]

Fuzzy numbers	(x_1, x_2, x_3)
$\tilde{1}$	(1, 1, 2)
$\tilde{2}$	(1, 2, 3)
$\tilde{3}$	(2, 3, 4)
$\tilde{4}$	(3, 4, 5)
$\tilde{5}$	(4, 5, 6)
$\tilde{6}$	(5, 6, 7)
$\tilde{7}$	(6, 7, 8)
$\tilde{8}$	(7, 8, 9)
$\tilde{9}$	(8, 9, 9)

namely, *effective range*, *flight height, flight velocity, reliability*, *firing accuracy*, *destruction rate*, and *kill radius*, whereas for *economy,* there are three subcriteria, namely, *system cost*, *system life*, and *material limitation*. Each criterion and subcriterion also has its weight, as shown in the second column of Table I. The performances of the three companies for each subcriterion are also given in Table I. We observe the following.

1) The major criteria are not equally weighted but, instead, are weighted using fuzzy numbers[3] (T1 FSs, as depicted in Fig. 3 and Table II) in the following order of importance: *tactics, advancement, economy, technology,* and *maintenance*. These weightings were established ahead of time by the contractor and not by the companies.
2) *Tactics* has seven subcriteria, *technology* and *maintenance* each have five subcriteria, and *economy* and *advancement* each have three subcriteria; hence, there are 23 subcriteria, all of which were established ahead of time by the contractor and not by the companies.
3) All of the subcriteria are weighted using fuzzy numbers. These weightings have also been established ahead of time by the contractor and not by the companies, and have been established separately within each of the five criteria and not simultaneously across all of the 23 subcriteria.
4) The performance evaluations for all 23 subcriteria are shown for the three companies and are either numbers or words. Usually, the numerical scores are obtained by extensive simulations and a few live firings of the missiles [3], [15]. It is not clear how the linguistic scores were obtained, so it is speculated that the contractor provided them based on other evidence and perhaps on some subjective rules.

[3]It is a common practice to use a tilde overmark to denote a fuzzy number that is modeled using a T1 FS. Even though it is also a common practice to use such a tilde overmark to denote an IT2 FS, we shall not change this common practice for a fuzzy number in this paper. Instead, we shall indicate in the text when the fuzzy number $\tilde{n}$ is modeled either as a T1 or as an IT2 FS.

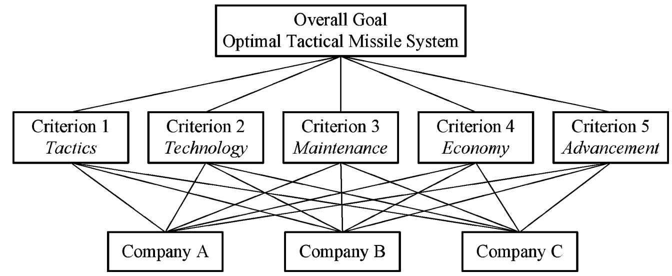

Fig. 4. Structure of evaluating competing tactical missile systems from three companies [54].

5) How to aggregate all of this data seems like a daunting task, especially since it involves numbers, fuzzy numbers for the weights, and words.
6) Finally, we believe that there should be an uncertainty band for each numerical score (except missile scale, which should be certain once the design is finished) because the numbers correspond to measurements of physical properties obtained from an ensemble of test missiles [3], [15], [18]. These bands have not been provided, but will be assumed, in this paper, to inject some additional realism into this application. This is very important because according to Harvard Business Essentials [2, p. 59], "in business, uncertainty of outcome is synonymous with risk, and you must factor it into your evaluation." We believe this is very necessary in weapon evaluations.

The missile-evaluation problem can also be summarized by Fig. 4. It is very clear from this figure that this is a multicriteria and two-level decision-making problem. At the first level, each of the three companies[4] is evaluated for its performance on five criteria: *tactics*, *technology*, *maintenance*, *economy,* and *advancement*. The lines emanating from each of the companies to these criteria indicate these evaluations, each of which involves a number of important (but not shown) subcriteria and their weighted aggregations that are described shortly. The second level in this hierarchical decision-making problem involves a weighted aggregation of the five criteria for each of the three companies.

How the Per-C will be used to assist a decision maker to decide which of the three companies is the winner in this procurement competition is explained in Section IV. However, first, we explain the CWW Engine that will be used in the Per-C. It is called an *NWA*.

III. NOVEL WEIGHTED AVERAGES

The *weighted average* (WA)

$$y = \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i} \tag{6}$$

is arguably the earliest and still most widely used form of aggregation or fusion, in which w_i are the weights (real numbers) that act upon the subcriteria x_i (real numbers).

[4]The terms *company* and *system* are used interchangeably in this paper.

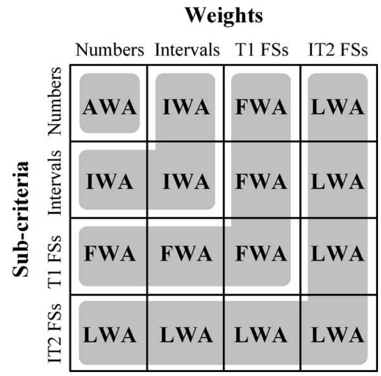

Fig. 5. Matrix of possibilities for a WA.

The arithmetic WA (AWA) is the one we are all familiar with and is the one in which all subcriteria and weights in (6) are real numbers. In many situations, however, providing a single number for either the subcriteria or weights is problematic (there could be uncertainties about them), and it is more meaningful to provide intervals T1 FSs, IT2 FSs [43], or a mixture of all of these for the subcriteria and weights.

Definition 1: An NWA is a WA in which at least one subcriterion or weight is not a single real number, but instead, is an interval, a T1 FS or an IT2 FS, in which case, such subcriteria or weights are called *novel models*.

Because there can be four possible models for subcriteria or weights, there can be 16 different WAs, as summarized in Fig. 5.

Definition 2: When at least one subcriterion or weight is modeled as an interval, and all other subcriteria or weights are modeled by no more than such a model, the resulting WA is called an *interval WA* (IWA).

Definition 3: When at least one subcriterion or weight is modeled as a T1 FS, and all other subcriteria or weights are modeled by no more than such a model, the resulting WA is called a *fuzzy WA* (FWA).

Definition 4: When at least one subcriterion or weight is modeled as an IT2 FS, the resulting WA is called a *linguistic WA* (LWA).

Definition 1 (Continued): By an NWA, we meant an IWA, an FWA, or an LWA.

Because NWAs are used as our CWW engine, more discussions about the IWA, FWA, and LWA, including how to compute them, are given in the Appendix.

IV. Missile Evaluation: A Perceptual Computing Approach

Recall that the Per-C has three components: encoder, CWW engine, and decoder. When perceptual computing is used for the missile-evaluation problem, each of these components must be considered.

A. Encoder

In this application, mixed data are used—crisp numbers, T1 fuzzy numbers, and words. The codebook contains the crisp numbers, the T1 fuzzy numbers with their associated T1 FS models (see Fig. 3 and Table II), and the words and their IT2 FS models.

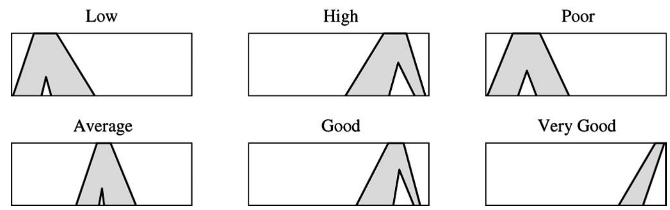

Fig. 6. IT2 FS models for the six words used in missile evaluation.

To ensure that LWAs are not unduly influenced by large numbers, all of the Table I numbers were mapped into [0, 10]. Let x_1, x_2, and x_3 denote the raw numbers for companies A, B, and C, respectively. For the 13 subcriteria whose inputs are numbers, these raw numbers were transformed into

$$x_i \rightarrow x_i' = \frac{10x_i}{\max(x_1, x_2, x_3)}. \tag{7}$$

Examining Table I, we observed that the words used for the remaining ten subcriteria are $\{$*low, high*$\}$ and $\{$*poor, average, good, very good*$\}$. Because this application is being used merely to illustrate how a Per-C can be used for missile system evaluation, and we do not have access to domain experts, interval-endpoint data were not collected for these words in the context of this application. Instead, each word was mapped into a synonym in Fig. 1:

$$\left.\begin{array}{lcl}\text{Low} & \rightarrow & \text{Low Amount}\\ \text{High} & \rightarrow & \text{High Amount}\end{array}\right\} \tag{8}$$

$$\left.\begin{array}{lcl}\text{Poor} & \rightarrow & \text{Small}\\ \text{Average} & \rightarrow & \text{Medium}\\ \text{Good} & \rightarrow & \text{Large}\\ \text{Very Good} & \rightarrow & \text{Very Large}\end{array}\right\}. \tag{9}$$

The IT2 FS models of the six words are shown in Fig. 6.

From Table I, we observed that some subcriteria may have a positive connotation and others may have a negative connotation. The following six subcriteria have a negative connotation.

1) *Flight height*: The lower the flight height, the better, because it is then more difficult for a missile to be detected by radar.
2) *Missile scale*: A smaller missile is harder to detect by radar.
3) *Reaction time*: A missile with shorter reaction time can respond more quickly.
4) *System cost*: The cheaper, the better.
5) *Operation condition requirement*: A missile with lower operation condition requirement can be deployed more easily and widely.
6) *Material limitation*: A missile with lower material limitation can be produced more easily, especially during wartime.

The first four of these subcriteria have numbers as their inputs. For them, a preprocessing step is needed to convert a large x_i' into a small number x_i^* and a small x_i' into a large number x_i^*, i.e.,

$$x_i \to x_i^* = 1/x_i \tag{10}$$

and then, (7) is applied to x_i^*, i.e.,

$$x_i^* \to x_i' = \frac{10x_i^*}{\max(x_1^*, x_2^*, x_3^*)}. \tag{11}$$

Equations (10) and (11) can be summarized into one equation as

$$x_i \to x' = \frac{10\min(x_1, x_2, x_3)}{x_i}. \tag{12}$$

Example 1: Suppose that $x_1 = 3$, $x_2 = 4$, and $x_3 = 5$. Then, when these numbers are mapped into $[0, 10]$ using (7), they become: $x_1' = 10(3/5) = 6$, $x_2' = 10(4/5) = 8$, and $x_3' = 10(5/5) = 10$. On the other hand, for subcriteria with negative connotation, these numbers are mapped into $[0, 10]$ using (12), and they become $x_1' = 10(3/3) = 10$, $x_2' = 10(3/4) = 7.5$, and $x_3' = 10(3/5) = 6$.

For the other two subcriteria with a negative connotation (operation condition requirement, and material limitation), *antonyms* [30], [57], [61], [91] are used for the words in (8) and (9), i.e.,

$$\mu_{10-A}(x) = \mu_A(10 - x)\ \forall x \tag{13}$$

where $10 - A$ is the antonym of a T1 FS A, and 10 is the right end of the domain of all FSs used in this paper. The definition in (13) can be easily extended to IT2 FSs, i.e.,

$$\mu_{10-\tilde{A}}(x) = \mu_{\tilde{A}}(10 - x)\ \forall x \tag{14}$$

where $10 - \tilde{A}$ is the antonym of an IT2 FS $\tilde{A}$. Because an IT2 FS is completely characterized by its LMF and UMF, each of which is a T1 FS, $\mu_{10-\tilde{A}}(x)$ in (14) is obtained by applying (13) to both $\underline{A}$ and $\overline{A}$.

Comment: Using these mappings, the highest score for the numerical subcriteria that have a positive connotation is always assigned the value 10, and the lowest score for the numerical subcriteria that have a negative connotation is also always assigned the value 10. What if such scores are not actually "good" scores? Assigning it, our highest value does not then seem to be correct.

In this type of procurement competition, the contractor often sets specifications on numerical performance subcriteria. Unfortunately, such specifications do not appear in any of the published articles about this application; therefore, we have had to do the best we can without them. If, for example, the contractor had set a specification for *reliability* as at least 85%, then (see Table I) no company should get a 10. A different kind of normalization would then have to be used.

B. Computing With Word Engine

The CWW engine is used to aggregate the criteria and subcriteria and, hence, to obtain the overall performance of each missile system. There are many different aggregation operators (functions) [5], [7], e.g., Beliakov *et al.* [5] classify them into four categories: 1) averaging functions; 2) conjunctive functions; 3) disjunctive functions; and 4) mixed functions. The averaging functions are the most frequently used aggregation operators in decision making. Among them, there are additive operators, e.g., the NWAs and the ordered WAs (OWAs) [83], and nonadditive operators, e.g., the Choquet integral [14] and the Sugeno integral [55]. Additive operators are defined on additive measures, i.e., [5], "*the measure of a set is the sum of the measures of its nonintersecting subsets*," whereas nonadditive operators are defined on nonadditive measures, i.e., [5], "*the measure of the total can be larger or smaller than the sum of the measures of its components*."

Choquet and Sugeno integrals are used to model the interactions among the inputs. Take the Choquet integral for example. As pointed out by Beliakov *et al.* [5], "*the main purpose of Choquet integral-based aggregation is to combine the inputs in such a way that not only the importance of individual inputs (as in weighted means), or of their magnitude (as in OWA), are taken into account, but also of their groups (or coalitions). For example, a particular input may not be important by itself, but becomes very important in the presence of some other inputs*." This kind of aggregation is very suitable for applications such as medical diagnosis because [5] "*some symptoms by themselves may not be really important, but may become key factors in the presence of other signs*." However, in the missile-evaluation problem, the criteria and subcriteria are nonintersecting; hence, there is no need to model the interactions among the inputs, and is why NWAs are used as our CWW engine.

During the aggregation, each of the major criteria had an NWA computed for it. Examining Table I, we observed that the NWA for *tactics* ($\tilde{Y}_1$) is an FWA (because the weights are T1 FSs and the subcriteria evaluations are numbers), whereas the NWAs for *technology* ($\tilde{Y}_2$), *maintenance* ($\tilde{Y}_3$), *economy* ($\tilde{Y}_4$), and *advancement* ($\tilde{Y}_5$) are LWAs (because at least one subcriterion evaluation is a word modeled by an IT2 FS). More specifically

$$\tilde{Y}_1 = \frac{\sum_{i=1}^{7} X_i W_i}{\sum_{i=1}^{7} W_i} \tag{15}$$

$$\tilde{Y}_2 = \frac{\sum_{i=8}^{12} \tilde{X}_i \tilde{W}_i}{\sum_{i=8}^{12} \tilde{W}_i} \tag{16}$$

$$\tilde{Y}_3 = \frac{\sum_{i=13}^{17} \tilde{X}_i \tilde{W}_i}{\sum_{i=13}^{17} \tilde{W}_i} \tag{17}$$

$$\tilde{Y}_4 = \frac{\sum_{i=18}^{20} \tilde{X}_i \tilde{W}_i}{\sum_{i=18}^{20} \tilde{W}_i} \tag{18}$$

$$\tilde{Y}_5 = \frac{\sum_{i=21}^{23} \tilde{X}_i \tilde{W}_i}{\sum_{i=21}^{23} \tilde{W}_i}. \tag{19}$$

These six NWAs are then aggregated by another NWA to obtain the overall performance, i.e., $\tilde{Y}$, as follows:

$$\tilde{Y} = \frac{\tilde{9}\tilde{Y}_1 + \tilde{3}\tilde{Y}_2 + \tilde{1}\tilde{Y}_3 + \tilde{5}\tilde{Y}_4 + \tilde{7}\tilde{Y}_5}{\tilde{9} + \tilde{3} + \tilde{1} + \tilde{5} + \tilde{7}}. \tag{20}$$

As a reminder to the reader, when $i = 2, 8, 9, 13, 18,$ and 20, (10) or the antonyms of the corresponding IT2 FSs must be used in (15)–(18).

C. Decoder

Our decoder computes ranking, similarity, and centroid. Rankings of the three companies are obtained for the six LWA FOUs in (15)–(20) using a centroid-based ranking method [76], which ranks IT2 FSs based on their average centroids (see Definition 6 in the Appendix). The average centroids for companies A, B, and C are represented in all figures in Section IV-D by $*$, $\diamond$, and $\circ$, respectively.

The Jaccard similarity in (5) is computed only for the three companies' overall performances $\tilde{Y}$ so that one can observe how similar the overall performances are for them.

Centroids are also computed for the three companies' $\tilde{Y}$, and provide a measure of uncertainty [52], [74] for each company's overall ranking, since $\tilde{Y}$ has propagated both numerical and linguistic uncertainties through their calculations.

D. Examples

This section contains examples that illustrate the missile-evaluation results for different scenarios. Example 2 uses the data that are in Table I as they are. Examples 4 and 5 use intervals for all numerical values (Example 3 explains how such intervals can be normalized), i.e., in Example 4, each numerical value x (except missile scale, which has no uncertainty) is changed to the interval $[x - 10\%x, x + 10\%x]$ for all three companies, and in Example 5, x is changed to $[x - 10\%x, x + 10\%x]$ for company A, $[x - 20\%x, x + 20\%x]$ for company B, and $[x - 5\%x, x + 5\%x]$ for company C. The use of more realistic data intervals instead of numbers is something that was mentioned earlier in Section II in Item 6.

Example 2: As just mentioned, this example uses the data that are in Table I as they are. In all figures, system A is represented by the solid curve, system B is represented by the dashed curve, and system C is represented by the dotted curve. In order to simplify the notation in the figures, the notations $\tilde{Y}_{Aj}$, $\tilde{Y}_{Bj}$, and $\tilde{Y}_{Cj}$ are used for aggregated results for criterion j and for companies A, B, and C, respectively. The caption of each figure indicates the name of criterion j ($j = 1, 2, \ldots, 5$), and the numbering of the criteria corresponds to their numbering in Table I.

FOUs for *tactics*, *technology*, *maintenance*, *economy,* and *advancement* are depicted in Fig. 7(b)–(e), respectively. FOUs for *overall performance* are depicted in Fig. 7(f). From Fig. 7(f), we observed that not only is FOU($\tilde{Y}_B$) visually well to the right of the other two FOUs, but its average centroid (which is on the horizontal axis) is also well to the right of those for companies A and C. Therefore, on the basis of ranking alone, company B would be declared the winner. This happens because system B ranks first in *maintenance*, *economy,* and *advancement,* and by significant amounts. Although it ranks last for *tactics* and *technology*, its FOUs for these two criteria are very close to those of systems A and C.

Table III summarizes the similarities between $\tilde{Y}_A$, $\tilde{Y}_B$, and $\tilde{Y}_C$. Observe that $\tilde{Y}_B$ is not very similar to either $\tilde{Y}_A$ or $\tilde{Y}_C$, so choosing company B as the winner is further reinforced, i.e., it is not a close call.

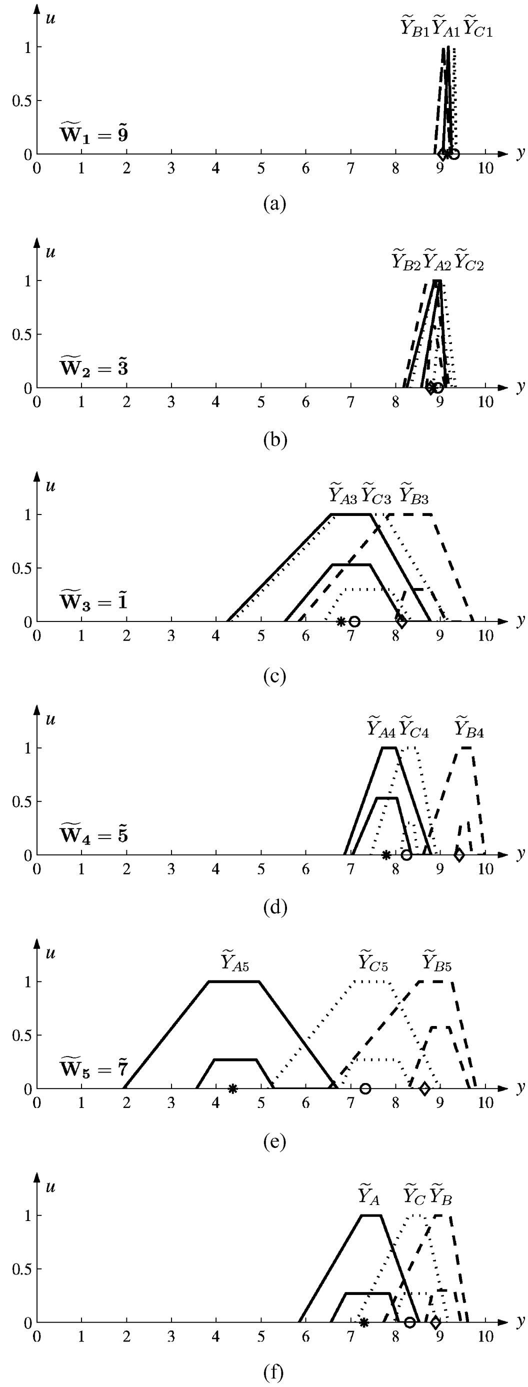

Fig. 7. Example 2. Aggregation results for (a) criterion 1: *tactics*; (b) criterion 2: *technology*; (c) criterion 3: *maintenance*; (d) criterion 4: *economy*; (e) criterion 5: *advancement*; and (f) overall performances of the three systems. The average centroids for companies A, B, and C are shown in all figures by $*$, $\diamond$, and $\circ$, respectively. The FOUs in (b)–(f) are not filled in; therefore, the three IT2 FSs can be distinguished more easily.

TABLE III
SIMILARITIES OF $\tilde{Y}$ IN EXAMPLE 2 FOR THE THREE COMPANIES

Company	$\tilde{Y}_A$	$\tilde{Y}_B$	$\tilde{Y}_C$
$\tilde{Y}_A$	1	0.05	0.18
$\tilde{Y}_B$	0.05	1	0.31
$\tilde{Y}_C$	0.18	0.31	1

TABLE IV
CENTROIDS, CENTERS OF CENTROID, AND RANKING BANDS OF $\tilde{Y}$ FOR VARIOUS UNCERTAINTIES

System		0% for all three systems Example 2	±10% for all three systems Example 4	±10% for System A ±20% for System B and ±5% for System C Example 5
A	C_A	[6.92, 7.67]	[6.59, 7.53]	[6.48, 7.42]
	c_A	7.30	7.06	6.95
	r_A	7.30±0.37	7.06±0.47	6.95±0.47
B	C_B	[8.59, 9.19]	[8.24, 8.99]	[7.93, 8.83]
	c_B	8.89	8.61	8.38
	r_B	8.89±0.30	8.61±0.37	8.38±0.45
C	C_C	[7.99, 8.65]	[7.70, 8.58]	[7.66, 8.42]
	c_C	8.32	8.14	8.04
	r_C	8.32±0.33	8.14±0.44	8.04±0.38

Finally, the centroids of $\tilde{Y}_A$, $\tilde{Y}_B$, and $\tilde{Y}_C$ (see Table IV) are $C_A = [6.92, 7.67]$, $C_B = [8.59, 9.19]$, and $C_C = [7.99, 8.65]$. Let the numerical rankings be the average centroids [see (A26) in the Appendix]. It follows that $c_A = 7.30$, $c_B = 8.89$, and $c_C = 8.32$; the half-lengths of each centroid [see (A27) in the Appendix] are $\delta_A = 0.37$, $\delta_B = 0.30$, and $\delta_C = 0.33$. One way to use these half-lengths is to summarize the rankings as $r = c \pm \delta$, i.e., $r_B = 8.89 \pm 0.30$, $r_C = 8.32 \pm 0.33$, and $r_A = 7.30 \pm 0.37$. Note that the centroids can also be interpreted as *ranking bands* [52] and that there is no or little overlap of these bands in this example. All these results are summarized in Table IV.

Not only does company B have the largest ranking, but it also has the smallest uncertainty band about that ranking, and $\tilde{Y}_B$ is not very similar to either $\tilde{Y}_A$ or $\tilde{Y}_C$. Choosing company B as the winner seems the right thing to do. This decision is also consistent with those obtained in [10], [11], [13], and [54].

As we have explained in Item 6 of Section II, in reality, there are uncertainties about each of the numbers in Table I, except missile scale, which is fixed once the missile design is finished. In the remaining examples, uncertainty intervals are assigned to each of these numbers, except missile scale, so that the effects of such uncertainties on the overall performances of the three companies can be studied.

For the ten subcriteria that have a positive connotation, the uncertainty intervals are

$$\begin{aligned} x_i &\rightarrow [x_i - v\%x_i, \min(x_i + v\%x_i, \max(x_1, x_2, x_3))] \\ &\equiv [\alpha_i, \beta_i], \quad i = 1, 2, 3. \end{aligned} \tag{21}$$

Note that $\max(x_1, x_2, x_3)$ is used as an upper limit, so that the converted number is not larger than 10 [see (22)]. The specific choice(s) made for v are explained in the examples. Equation (7) is then used for the two endpoints in (21), i.e.,

$$[\alpha_i, \beta_i] \rightarrow \left[\frac{10\alpha_i}{\max(\beta_1, \beta_2, \beta_3)}, \frac{10\beta_i}{\max(\beta_1, \beta_2, \beta_3)}\right]. \tag{22}$$

For the two subcriteria (flight height and reaction time) that have a negative connotation, the uncertainty intervals are

$$\begin{aligned} x_i &\rightarrow [\max(x_i - v\%x_i, \min(x_1, x_2, x_3)), x_i + v\%x_i)] \\ &\equiv [\alpha_i', \beta_i'], \quad i = 1, 2, 3. \end{aligned} \tag{23}$$

Note that $\min(x_1, x_2, x_3)$ is used as a lower limit; therefore, the converted number is not larger than 10 [see (24)]. Equation (12) is then used for the two endpoints in (23) so that

$$[\alpha_i', \beta_i'] \rightarrow \left[\frac{10\min(\alpha_1', \alpha_2', \alpha_3')}{\beta_i'}, \frac{10\min(\alpha_1', \alpha_2', \alpha_3')}{\alpha_i'}\right]. \tag{24}$$

The following example illustrates (21)–(24).

Example 3: As in Example 1, suppose that $x_1 = 3$, $x_2 = 4$, and $x_3 = 5$. Let $v = 10$. For a subcriterion with positive connotation, it follows from (21) that $x_1 \rightarrow [2.7, 3.3]$, $x_2 \rightarrow [3.6, 4.4]$, and $x_3 \rightarrow [4.5, 5]$. Using (22), one finds that

$$[2.7, 3.3] \rightarrow [10(2.7/5), 10(3.3/5)] = [5.4, 6.6]$$
$$[3.6, 4.4] \rightarrow [10(3.6/5), 10(4.4/5)] = [7.2, 8.8]$$
$$[4.5, 5] \rightarrow [10(4.5/5), 10(5/5)] = [9, 10].$$

For a subcriteria with negative connotation, it follows from (23) that $x_1 \rightarrow [3, 3.3]$, $x_2 \rightarrow [3.6, 4.4]$, and $x_3 \rightarrow [4.5, 5.5]$. Using (24), one finds that

$$[3, 3.3] \rightarrow [10(3/3.3), 10(3/3)] = [9.1, 10]$$
$$[3.6, 4.4] \rightarrow [10(3/4.4), 10(3/3.6)] = [6.8, 8.3]$$
$$[4.5, 5.5] \rightarrow [10(3/5.5), 10(3/4.5)] = [5.5, 6.7].$$

Example 4: In this example, each numerical value x in Table I, except missile scale, is changed by the same percentage amount to the interval $[x - 10\%x, x + 10\%x]$. We are interested to learn if such uncertainty intervals change the rankings of the three companies. FOUs for *tactics*, *technology*, *maintenance*, *economy,* and *advancement* are depicted in Fig. 8(a)–(e), respectively. The overall performances of the three systems are depicted in Fig. 8(f). System B still appears to be the winning system.

Comparing the results shown in Fig. 8 with their counterparts shown in Fig. 7, we observed that the FOUs have larger support generally. Particularly, the T1 FSs shown in Fig. 7(b) are triangular, whereas the T1 FSs shown in Fig. 8(a) are trapezoidal. This is because in Fig. 7(b), the inputs to the subcriteria are numbers and the weights are triangular T1 FSs, and hence, the $\alpha = 1$ α-cut on $\tilde{Y}_{A1}$ ($\tilde{Y}_{B1}$, or $\tilde{Y}_{C1}$) is an AWA, whereas in Fig. 8(a), the inputs to the subcriteria are intervals and the weights are triangular T1 FSs, and hence, the $\alpha = 1$ α-cut on $\tilde{Y}_{A1}$ ($\tilde{Y}_{B1}$, or $\tilde{Y}_{C1}$) is an IWA.

Table V summarizes the similarities among $\tilde{Y}_A$, $\tilde{Y}_B$, and $\tilde{Y}_C$. We observed that $\tilde{Y}_C$ is much more similar to $\tilde{Y}_B$ in this example than it was in Example 2. Consequently, one may be less certain about choosing company B as the winner when there is ±10% uncertainty on all of the numbers in Table I than when there is no uncertainty on those numbers.

The centroids, centers of centroids, and the ranking bands of $\tilde{Y}_A$, $\tilde{Y}_B$, and $\tilde{Y}_C$ are shown in Table IV. We observed that not only does company B still have the largest ranking, but it still also has the smallest uncertainty band about that ranking. However, when there is ±10% uncertainty on all of the numbers in Table I, not only do the numerical rankings for the three companies shift to the left (to lower values), but the uncertainty bands about

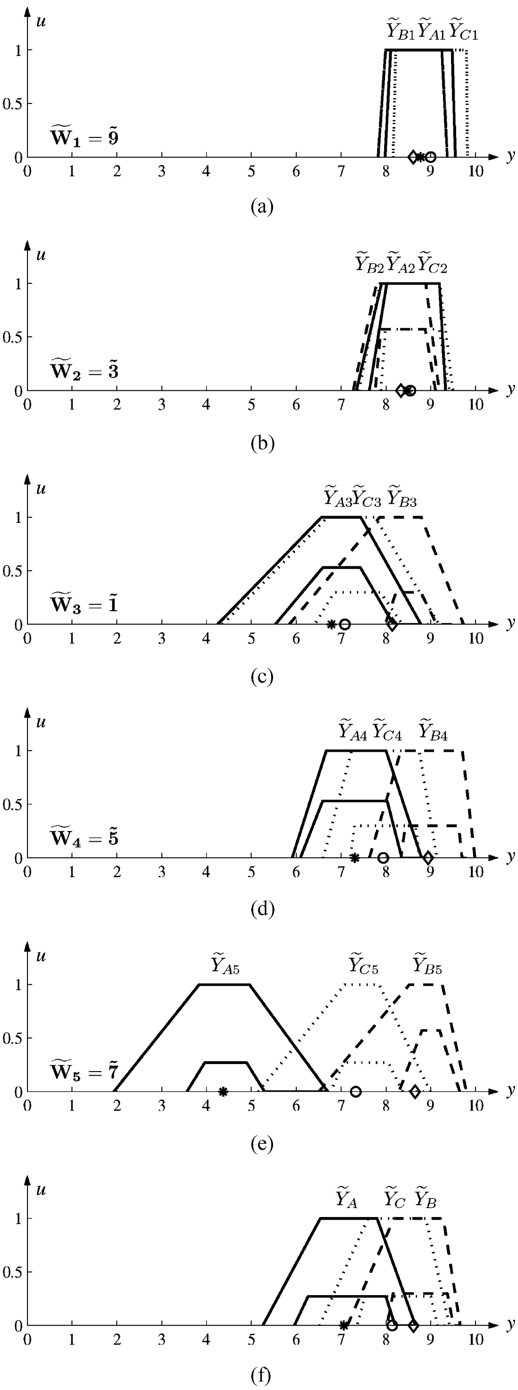

Fig. 8. Example 4. Aggregation results for (a) criterion 1: *tactics*; (b) criterion 2: *technology*; (c) criterion 3: *maintenance*; (d) criterion 4: *economy*; (e) criterion 5: *advancement*; and (f) overall performances of the three systems. The average centroids for companies A, B, and C are shown in all figures by $*$, $\diamond$, and $\circ$, respectively.

these rankings also increase. The overlap between the ranking bands of systems B and C also increases.

In short, even though company B could still be declared the winner, one is less certain about doing this when there is $\pm 10\%$ uncertainty on all of the numbers in Table I.

Example 5: In this example, each numerical value x in Table I is changed to $[x - 10\%x, x + 10\%x]$ for company

TABLE V
SIMILARITIES OF $\tilde{Y}$ IN EXAMPLE 4 FOR THE THREE COMPANIES

Company	$\tilde{Y}_A$	$\tilde{Y}_B$	$\tilde{Y}_C$
$\tilde{Y}_A$	1	0.14	0.33
$\tilde{Y}_B$	0.14	1	0.59
$\tilde{Y}_C$	0.33	0.59	1

A, $[x - 20\%x, x + 20\%x]$ for company B, and $[x - 5\%x, x + 5\%x]$ for company C. FOUs for *tactics*, *technology*, *maintenance*, *economy,* and *advancement* are depicted in Fig. 9(a)–(e), respectively. The overall performances of the three systems are depicted in Fig. 9(f). We observed that the UMF (LMF) of $\tilde{Y}_C$ is completely inside the UMF (LMF) of $\tilde{Y}_B$; therefore, it is difficult to declare system B the winner.

Table VI summarizes the similarities among $\tilde{Y}_A$, $\tilde{Y}_B$, and $\tilde{Y}_C$. We observed that $\tilde{Y}_C$ is more similar to $\tilde{Y}_B$ in this example than in Example 4; therefore, one may be less certain about choosing company B as the winner in this case.

The centroids, centers of centroids, and the ranking bands of $\tilde{Y}_A$, $\tilde{Y}_B$, and $\tilde{Y}_C$ are shown in Table IV. Now the ranking bands for systems B and C overlap a lot, which is why it is difficult to declare system B the winner.

This example clearly demonstrates that providing only average values for the subcriteria in Table I can lead to misleading conclusions. Uncertainty bands about these average values can change conclusions dramatically.

In summary, our Per-C approach can consider scenarios with different levels of uncertainties, and hence evaluate the robustness of the final decision, e.g., the numerical rankings of the three companies (c) when v [see (21) and (23)] changes by the same amount for all three systems from 0 to 20 are shown in Fig. 10(a), and the corresponding half-length of the ranking bands (δ) are shown in Fig. 10(b). Observe that Company B is always the best choice, but as v increases, c decreases, and δ increases, which means that the ranking band overlap between $\tilde{Y}_B$ and $\tilde{Y}_C$ increases, and hence, the lead of company B over company C decreases.

V. Comparisons With Previous Approaches

In this section, our Per-C approach is compared with four previous approaches to the missile-evaluation problem. We are only able to do this for the 0% uncertainty situation of Example 2, because none of the previous methods were developed to handle intervals of numbers for the subcriteria.

Note that the missile systems evaluation problem is quite different from other applications of fuzzy-logic systems, e.g., fuzzy-logic control [8], [22], [23], [39], [41], [60], [80], [81], [95], where well-established measures can be used to quantify the performance for validation. In decision making, usually there is no ground truth data or quantitative measures to assess the performance of a method. The Per-C for missile evaluation provides aids to the decision maker, who then uses them to make the final decision. This is why "plausibility" is used rather than "validation."

We believe that for a decision to be plausible, the decision-making process must be reasonable and transparent. Therefore, in the following, we summarize each of the four previous

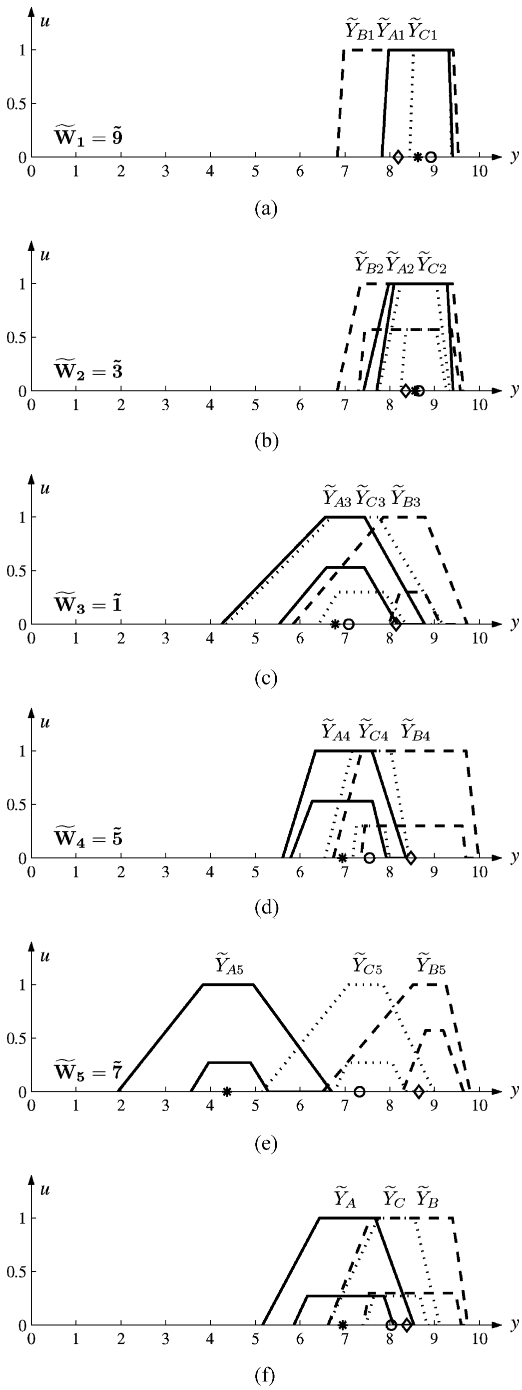

Fig. 9. Example 5. Aggregation results for (a) criterion 1: *tactics*; (b) criterion 2: t; (c) criterion 3: *maintenance*; (d) criterion 4: *economy*; (e) criterion 5: *advancement*; and (f) overall performances of the three systems. The average centroids for companies A, B, and C are shown in all figures by $*$, $\diamond$, and $\circ$, respectively.

approaches and point out its limitations. From the comparative analysis, the reader should be able to conclude that these four approaches have obvious limitations, and our Per-C approach is the most reasonable. As a result, we believe our result is also the most plausible.

Verification of a decision can only occur after (and sometimes long after) the decision has been made when the consequences of the decisions can be observed and evaluated. As a result, it is not possible to verify different approaches when each approach provides only an aid to the decision maker.

TABLE VI
SIMILARITIES OF $\widetilde{Y}$ IN EXAMPLE 5 FOR THE THREE COMPANIES

System	$\widetilde{Y}_A$	$\widetilde{Y}_B$	$\widetilde{Y}_C$
$\widetilde{Y}_A$	1	0.24	0.27
$\widetilde{Y}_B$	0.24	1	0.64
$\widetilde{Y}_C$	0.27	0.64	1

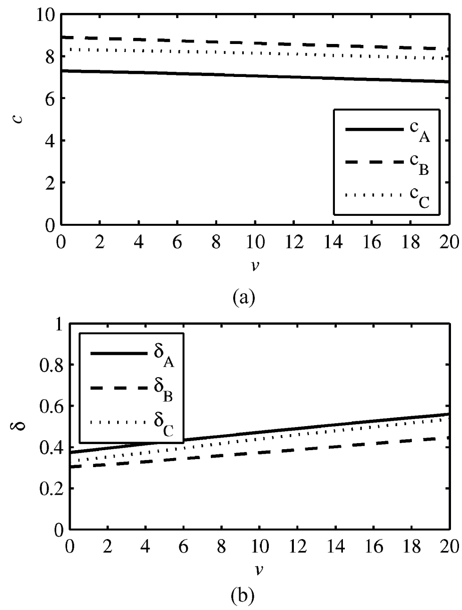

Fig. 10. (a) Ranking of the three companies, i.e., c, and (b) half-length of ranking band, i.e., δ, when v changes from 0 to 20.

A. Comparison With Mon et al.'s Approach

Mon *et al.* [54] appear to be the first to work on "*performance evaluation and optimal design of weapon systems [as] multiple criteria decision-making problems*" using FSs. They perform the following steps (we comment on some of these steps next):

1) *Convert each subcriterion entry in Table I into either 1 if the (contractor's specified) subcriterion is satisfied or 0.5 or 0 if the subcriterion is not satisfied.*[5]
2) *Aggregate the subcriteria crisp scores by first adding them* (implying that they are given the same weight) *to provide a total score, and then, mapping it into a fuzzy number. The fuzzy numbers are then put into a* 3×5 fuzzy-judgment matrix X. An example of computing the total scores for tactics and maintenance, based on the entries in Table I, is shown in Table VII, and[6] the fuzzy-judgment matrix is given in (25), as shown at the bottom of the next page.

[5]For tactics, the following specifications are given [54]: "If the *effective range* is further than 40 km, the *flight height* is smaller than 20 m, the *flight velocity* is greater than 0.8 Mach number, *reliability*is greater than 80%, *firing accuracy* is greater than 65%, *destruction rate* is greater than 85%, and *kill radius* is greater than 15 m, then the corresponding score of the [each] subcriterion is 1; otherwise, the score is 0.5." For maintenance, the following specifications are given [54]: "If the subcriteria are *good* or higher, their scores are 1; if they are *poor,* the score is 0; otherwise, the scores are 0.5." Specifications are not provided in [54] for technology, economy, and advancement.

[6]There is no obvious connection between the total scores in Table VII and their fuzzy counterparts in (25), which is something that we comment upon later in this section.

TABLE VII
MON ET AL.'S [54] SCORES AND TOTAL SCORES FOR THE TACTICS AND MAINTENANCE CRITERIA FOR THE THREE MISSILE SYSTEMS

	Tactics				Maintenance		
System	A	B	C	System	A	B	C
Effective range	1	0.5	0.5	Operation condition requirement	1	0.5	0.5
Flight height	0.5	1	0.5				
Flight velocity	0.5	1	0.5				
Reliability	1	1	0.5	Safety	1	0.5	0.5
Firing accuracy	1	1	0.5	Defilade	0.5	1	0.5
Destruction rate	0.5	1	1	Simplicity	0.5	0.5	0.5
Kill radius	1	0.5	1	Assembly	0.5	0.5	0
Total score	5.5	6	4.5	Total score	3.5	3	2

3) *Assign fuzzy importance weights (fuzzy numbers) to each of the five criteria*. These weights are indicated in Table I.
4) *Compute a* total fuzzy-judgment matrix Y *by multiplying each element of* X *by its fuzzy weight*, as in (26), shown at the bottom of the page. The multiplications are performed using α-cuts, for many values of α, the result for each α being Y_α, shown in (27) at the bottom of the page. Each element of Y_α is an interval $(y_{il})_\alpha = [(y_{il}^l)_\alpha, (y_{il}^r)_\alpha]$ ($i = 1, 2, 3$ and $l = 1, \ldots, 5$).
5) *Construct a* crisp judgment matrix $J(\alpha) = \{j_{ik}(\alpha)\}_{i=1,2,3,k=1,2,\ldots,5}$, where

$$j_{ik}(\alpha) = \lambda y_{ik}^l(\alpha) + (1-\lambda) y_{ik}^r(\alpha) \tag{28}$$

in which $\lambda \in [0, 1]$ *is called an* index of optimism *and*$j_{ik}(\alpha)$ *is called the* degree of satisfaction. When $j_{ik}(\alpha)$ is 0, or 1/2, or 1, the decision maker is called *pessimistic*, *moderate,* or *optimistic*, respectively. In [54], results are provide for these three values of λ.
6) *Compute the* entropy, i.e., e_i, *for each system*, as

$$e_i = -\sum_{k=1}^{5} f_{ik} \log_2(f_{ik}), \quad i = 1, 2, 3 \tag{29}$$

where

$$f_{ik} = \frac{j_{ik}(\alpha)}{\max_k j_{ik}(\alpha)}. \tag{30}$$

7) *Normalize the three entropies by diving each entropy number by the sum of the three entropy numbers, leading to three entropy weights, i.e., one for each company.* This is done for sampled values of $\alpha \in [0, 1]$ and for specified values of λ.
8) *Chooses the winning company as the one that has the largest entropy*.

Mon *et al.* [54] demonstrate that system B is the winner (which coincides with our results) and system A is better than system C (which does not agree with our results—see Table IV, column 3) for the three values of λ mentioned in Step 5, and for all values of α. Note the following in their approach.

1) Words are not modeled (probably because they did not know how to do this); instead, they are ranked in an *ad hoc* manner using the crisp numbers 0, 0.5, and 1.
2) When a numerical subcriterion is not satisfied, a company is assigned a score of 0.5 (or 0), regardless of how far away its score is from its specification; hence, useful information is lost.
3) Each subcriterion is weighted the same when the total score is computed, which is counterintuitive, and is very different from the weightings that are used in Table I.
4) How the "total score" is converted into a fuzzy number is not explained and seems very strange and inconsistent (e.g., for tactics, $5.5 \to \tilde{5}$, $6 \to \tilde{7}$, and $4.5 \to \tilde{1}$, and for maintenance, $3.5 \to \tilde{7}$, $3 \to \tilde{5}$, and $2 \to \tilde{1}$). Information is lost when this is done.
5) Step 6 is controversial since entropy is a measure of uncertainty [74] rather than a measure of overall performance of a system.
6) Their procedure has to be repeated for many values of α and λ. Although the same result was obtained in [54] for all values of α and λ, there is no guarantee that results could

$$X = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Tactics} & \text{Technology} & \text{Maintenance} & \text{Economy} & \text{Advancement} \end{array} \\ \begin{bmatrix} \tilde{5} & \tilde{1} & \tilde{7} & \tilde{3} & \tilde{1} \\ \tilde{7} & \tilde{5} & \tilde{5} & \tilde{5} & \tilde{7} \\ \tilde{1} & \tilde{3} & \tilde{1} & \tilde{1} & \tilde{5} \end{bmatrix} \end{array} \tag{25}$$

$$Y = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Tactics} & \text{Technology} & \text{Maintenance} & \text{Economy} & \text{Advancement} \end{array} \\ \begin{bmatrix} \tilde{5} \times \tilde{9} & \tilde{1} \times \tilde{3} & \tilde{7} \times \tilde{1} & \tilde{3} \times \tilde{5} & \tilde{1} \times \tilde{7} \\ \tilde{7} \times \tilde{9} & \tilde{5} \times \tilde{3} & \tilde{5} \times \tilde{1} & \tilde{5} \times \tilde{5} & \tilde{7} \times \tilde{7} \\ \tilde{1} \times \tilde{9} & \tilde{3} \times \tilde{3} & \tilde{1} \times \tilde{1} & \tilde{1} \times \tilde{5} & \tilde{5} \times \tilde{7} \end{bmatrix} \end{array} \tag{26}$$

$$Y_\alpha = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{Tactics} & \text{Technology} & \text{Maintenance} & \text{Economy} & \text{Advancement} \end{array} \\ \begin{bmatrix} (\tilde{5} \times \tilde{9})_\alpha & (\tilde{1} \times \tilde{3})_\alpha & (\tilde{7} \times \tilde{1})_\alpha & (\tilde{3} \times \tilde{5})_\alpha & (\tilde{1} \times \tilde{7})_\alpha \\ (\tilde{7} \times \tilde{9})_\alpha & (\tilde{5} \times \tilde{3})_\alpha & (\tilde{5} \times \tilde{1})_\alpha & (\tilde{5} \times \tilde{5})_\alpha & (\tilde{7} \times \tilde{7})_\alpha \\ (\tilde{1} \times \tilde{9})_\alpha & (\tilde{3} \times \tilde{3})_\alpha & (\tilde{1} \times \tilde{1})_\alpha & (\tilde{1} \times \tilde{5})_\alpha & (\tilde{5} \times \tilde{7})_\alpha \end{bmatrix} \end{array}. \tag{27}$$

TABLE VIII
CHEN'S [10] SCORES AND TOTAL SCORES FOR THE TACTICS AND MAINTENANCE CRITERIA FOR THE THREE MISSILE SYSTEMS

	Tactics				Maintenance		
System	A	B	C	System	A	B	C
Effective range	1	2	2	Operation condition requirement	1	2	2
Flight height	2	1	2				
Flight velocity	2	1	2				
Reliability	1	1	2	Safety	1	2	2
Firing accuracy	1	1	2	Defilade	2	1	2
Destruction rate	2	1	1	Simplicity	2	2	2
Kill radius	1	2	1	Assembly	2	2	3
Total score	10	9	12	Total score	8	9	11

not depend on both α and λ, in which case, conflicting conclusions could be reached.

In summary, although both Mon *et al.*'s approach and our Per-C approach conclude that system B is the best, we believe that this is more a coincidence because of the earlier significant differences. Mon *et al.* conclude that system A is better than system C, whereas we have the opposite conclusion. We believe that our result is more plausible because our Per-C approach has overcome Mon *et al.*'s limitations.

B. Comparison With Chen's First Approach

Chen has published two approaches [10], [11] for the same missile-evaluation problem. Chen [10] carried out the following steps.

1) *Converts each subcriterion entry in Table I into either 1 if the (contractor's specified) subcriterion is satisfied, or 2 or 3 if the subcriterion is not satisfied.*
2) *Aggregates the subcriteria crisp scores by first adding them* (implying that they are given the same weight) *to provide a total score, and then, mapping it into a fuzzy number. The fuzzy numbers are then put into a* 3×5 fuzzy rank score matrix[7] X. An example for computing the total scores of tactics and maintenance, on the basis of the entries in Table I, is shown in Table VIII, and the fuzzy rank score matrix is given in (31), shown at the bottom of the page. We observed that Table VIII is quite similar to Table VII, except that scores 0, 0.5, and 1 have been replaced by 3, 2, and 1, respectively, and (31) is analogous to (25).
3) *Assigns fuzzy importance weights (fuzzy numbers) to each of the five criteria*. These weights are indicated in Table I.
4) *Multiplies each element of X by its fuzzy importance weight, and then adds the resulting five fuzzy numbers for each company to obtain three (triangle) fuzzy numbers,*[8] R_A, R_B, *and* R_C. For triangle and trapezoidal fuzzy numbers (as in Table II), Chen explains how to do this without having to use α-cuts. His results are
$$R_A = (140, 199, 257)$$
$$R_B = (106, 159, 222)$$
$$R_C = (146, 208, 280). \quad (32)$$
5) *Defuzzifies each of these fuzzy numbers to obtain the rank of the three companies*, namely
$$\bar{R}_A = 198.75, \quad \bar{R}_B = 161.5, \quad \bar{R}_C = 210.5. \quad (33)$$
6) *Chooses the winning company as the one that has the smallest rank*. The smallest number is the winner because in [10], 1 is of higher rank than 2 or 3.

By this method, Chen arrives at the same results as Mon *et al.* do, namely, system B is the winner and system A is better than system C.

Comparing Chen's first approach with ours, we see the following:

1) Words are not modeled in Chen's first approach; instead, they are ranked in an *ad hoc* manner using the crisp numbers 3, 2, and 1.
2) It appears that Chen started wth Mon *et al.*'s scores and mapped 1 into 1, 0.5 into 2, and 0 into 3, regardless of how far away a score is from its specification; hence, useful information is lost again.
3) Each subcriterion is weighted the same when the total score is computed, which is counterintuitive, and is very different from the weightings that are used in Table I.

In summary, though Chen's first approach has some improvements over Mon *et al.*'s approach, e.g., how the "total score" is converted into a fuzzy number is transparent and it does not depend on α-cuts, it still has several limitations compared with our Per-C approach. Although both Chen's first approach and our Per-C approach conclude that system B is the best, we believe that this is more a coincidence because of the earlier significant differences. Chen concludes that system A is better than system C, whereas we have the opposite conclusion. We believe that our result is more plausible because our Per-C approach has overcome Chen's limitations.

C. Comparison With Chen's Second Approach

Chen [11] uses the index of optimism introduced in Mon *et al.*'s approach, i.e., he carried out the following steps.

1) *Assigns (for the first time) a* fuzzy importance number *to each of the subcriteria.*
2) *Ranks the subcriteria by using fuzzy ranking* ($\tilde{1}$, $\tilde{2}$, or $\tilde{3}$), where now $\tilde{3}$ is the highest rank and $\tilde{1}$ is the lowest rank. An example of this ranking for tactics and maintenance,

[7] This term is synonymous with Mon *et al.*'s [54] "fuzzy-judgment matrix."

[8] Recall that a triangle fuzzy number can be specified as (a, b, c), where a and c are its base endpoints, and b is its apex location.

$$\begin{array}{cc} & \begin{array}{ccccc} \text{Tactics} & \text{Technology} & \text{Maintenance} & \text{Economy} & \text{Advancement} \end{array} \\ X = \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{bmatrix} \tilde{10} & \tilde{9} & \tilde{8} & \tilde{5} & \tilde{7} \\ \tilde{9} & \tilde{7} & \tilde{9} & \tilde{4} & \tilde{4} \\ \tilde{12} & \tilde{8} & \tilde{11} & \tilde{6} & \tilde{5} \end{bmatrix} \end{array} \quad (31)$$

TABLE IX
CHEN'S [11] SCORES FOR THE TACTICS AND MAINTENANCE CRITERIA FOR THE THREE MISSILE SYSTEMS

System	Tactics A	Tactics B	Tactics C	System	Maintenance A	Maintenance B	Maintenance C
Effective range	$\tilde{3}$	$\tilde{1}$	$\tilde{2}$	Operation condition requirement	$\tilde{3}$	$\tilde{1}$	$\tilde{1}$
Flight height	$\tilde{1}$	$\tilde{3}$	$\tilde{2}$				
Flight velocity	$\tilde{1}$	$\tilde{3}$	$\tilde{2}$				
Reliability	$\tilde{2}$	$\tilde{3}$	$\tilde{1}$	Safety	$\tilde{2}$	$\tilde{1}$	$\tilde{1}$
Firing accuracy	$\tilde{2}$	$\tilde{3}$	$\tilde{1}$	Defilade	$\tilde{1}$	$\tilde{2}$	$\tilde{1}$
Destruction rate	$\tilde{1}$	$\tilde{3}$	$\tilde{2}$	Simplicity	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$
Kill radius	$\tilde{2}$	$\tilde{1}$	$\tilde{3}$	Assembly	$\tilde{2}$	$\tilde{2}$	$\tilde{1}$

on the basis of the entries in Table I that is consistent with these entries, is shown in Table IX.

3) *Computes a* fuzzy score *for each company by multiplying each subcriterion's fuzzy importance number by its fuzzy ranking, and then adding all of these products to obtain three (triangle) fuzzy numbers*, T_A, T_B, *and* T_C. His results are

$$\begin{aligned} T_A &= (134, 234, 418) \\ T_B &= (174, 276, 467) \\ T_C &= (125, 226, 412). \end{aligned} \tag{34}$$

4) *Computes the* α-cuts *of* T_A, T_B, *and* T_C, *for many values of* α, where the α-cuts of T_A, T_B, *and* T_C are denoted by $[a_1^{(\alpha)}, a_2^{(\alpha)}]$, $[b_1^{(\alpha)}, b_2^{(\alpha)}]$, and $[c_1^{(\alpha)}, c_2^{(\alpha)}]$, respectively.
5) *Lets* $\lambda \in [0, 1]$ *be an* index of optimism *and constructs the following three* crisp scores:

$$\begin{cases} D_a^{\lambda}(A|\alpha) = \lambda a_1^{(\alpha)} + (1-\lambda) a_2^{(\alpha)} \\ D_b^{\lambda}(B|\alpha) = \lambda b_1^{(\alpha)} + (1-\lambda) b_2^{(\alpha)} \\ D_c^{\lambda}(C|\alpha) = \lambda c_1^{(\alpha)} + (1-\lambda) c_2^{(\alpha)} \end{cases}. \tag{35}$$

6) *Normalizes these crisp scores to obtain*

$$\begin{cases} N_a^{\lambda}(A|\alpha) = D_a^{\lambda}(A|\alpha)/[D_a^{\lambda}(A|\alpha) + D_b^{\lambda}(B|\alpha) + D_c^{\lambda}(C|\alpha)] \\ N_b^{\lambda}(A|\alpha) = D_b^{\lambda}(A|\alpha)/[D_b^{\lambda}(A|\alpha) + D_b^{\lambda}(B|\alpha) + D_c^{\lambda}(C|\alpha)] \\ N_c^{\lambda}(A|\alpha) = D_c^{\lambda}(A|\alpha)/[D_a^{\lambda}(A|\alpha) + D_b^{\lambda}(B|\alpha) + D_c^{\lambda}(C|\alpha)]. \end{cases} \tag{36}$$

7) *Chooses the winning company as the one that has the largest normalized crisp score.*

Chen [11] demonstrates that system B is again the winner, and system A is again better than system C for $\lambda = 0$, 0.5, and 1 and for all values of α.

Comparing Chen's second approach with ours, we see the following.

1) Chen is still losing information by first assigning a fuzzy importance number to each subcriterion and then processing the ranked subcriteria.
2) In his step 3, Chen is weighting each of the five main criteria the same, which in counterintuitive.
3) As in Mon *et al.*'s approach [54], Chen's second approach has to be repeated for many values of α and λ, and although the same result was obtained in [11] for all values of α and λ, there is no guarantee that results could not depend on both α and λ, in which case, conflicting conclusions could again be reached.

In summary, though Chen improves Mon *et al.*'s approach by considering the different weights of the subcriteria, we still have doubts about his results because of the aforementioned three limitations.

D. Comparison With Cheng's Approach

Cheng proposes two approaches [12], [13] to evaluate the missile systems. Because his results are not consistent, we consider only his latest approach [13] in this paper, where he carried out the following steps.

1) *Converts each subcriterion entry in Table I into either 1, or 0.5 or 0*, as done in step 1 of Mon *et al.*'s approach.
2) *Computes a total score for each criterion*, as done in step 2 of Mon *et al.*'s approach, *maps it into a fuzzy number, and puts all these fuzzy numbers into a* 3×5 fuzzy-judgment matrix X, shown in[9] (37) at the bottom of the page.
3) *Assigns fuzzy importance weights to each of the five criteria*. These weights are indicated in Table I.
4) *Aggregates each system by computing the weighted sum of its fuzzy numbers*, e.g., for system A, it is

$$\begin{aligned} \tilde{r}_A &= \tilde{1} \otimes \tilde{9} \oplus \tilde{1} \otimes \tilde{3} \oplus \tilde{7} \otimes \tilde{1} \oplus \tilde{3} \otimes \tilde{5} \oplus \tilde{1} \otimes \tilde{7} \\ &= (21, 41, 131). \end{aligned} \tag{38}$$

5) *Ranks* $\tilde{r}_A$, $\tilde{r}_B$, *and* $\tilde{r}_C$ *to find the best system.*

Cheng obtains the results that system B is still the winner (which agrees with our result), but now, system C is better than system A (which also agrees with our result).

Comparing Cheng's approach with ours, we observe the following limitations of Cheng's approach.

1) Words are not modeled; instead, they are ranked in an *ad hoc* manner using the crisp numbers 0, 0.5, and 1.
2) When a numerical subcriterion is not satisfied, a company is assigned a score of 0.5 (or 0), regardless of how far away its score is from its specification; hence, useful information is lost.

[9] Observe that the first column of X in (37) is different from that in (25). How (37) was computed is unclear. The author only mentioned that [13] "assume that the tactical factors for 3 TMs can be increased by the opinions from experts . . ."

$$\begin{array}{cc} & \begin{array}{ccccc} \text{Tactics} & \text{Technology} & \text{Maintenance} & \text{Economy} & \text{Advancement} \end{array} \\ X = \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} & \begin{bmatrix} \tilde{1} & \tilde{1} & \tilde{7} & \tilde{3} & \tilde{1} \\ \tilde{7} & \tilde{5} & \tilde{5} & \tilde{5} & \tilde{7} \\ \tilde{9} & \tilde{3} & \tilde{1} & \tilde{1} & \tilde{5} \end{bmatrix} \end{array} \tag{37}$$

3) Each subcriterion is weighted the same when the total score is computed, which is counterintuitive, and is very different from the weightings that are used in Table I.
4) How the "total score" is converted into a fuzzy number is subjective and not explained.
5) The fuzzy number representing the overall performance of a system (e.g., $\tilde{r}_A$) is outside of [1, 9], the domain of the fuzzy numbers in Table II, because weighted sum instead of weighted average is used; Therefore, it is very difficult to assess how good a system is.

E. Summary

Four previous approaches on the same missile-evaluation problem have been introduced and compared with our Per-C approach. We observed the following points.

1) Although we do not agree with some steps in each of the four previous approaches, the Per-C approach reaches the same final decision as these four approaches, i.e., system B is the best. Since all four previous approaches were published in peer-reviewed journals, we believe that system B is indeed the best, i.e., our Per-C approach selects the best system correctly.
2) The four previous approaches have different conclusions in comparing the performance of systems A and C: the three earlier approaches (see Section V-A–V-C) suggested that system A is better than system C, whereas the latest approach (see Section V-D) suggested that System C is better than System A. Our Per-C approach gives the same result as the latest approach, i.e., system C is better than system A. We believe that Per-C again gives the right answer on this.

By the previous comparison, the meaningfulness and usefulness of Per-C is validated and verified. The discussions in this paper also suggest that Per-C is a very useful tool for multicriteria decision making, i.e., diverse information is correctly aggregated and uncertainties associated with the decision are given.

A reader may wonder why the Per-C is needed, given that Cheng's approach (see Section V-D) can reach the same conclusion. Recall that we have pointed out several limitations of Cheng's approach at the end of Section V-D. The Per-C approach avoids them because of its distinguishing features, which are listed next in our Section VI.

VI. Conclusion

The Per-C is an instantiation of Zadeh's CWW paradigm, as applied to assisting people in making subjective judgments. This paper has shown how the Per-C can be applied to a missile-evaluation problem, which is a hierarchical multicriteria decision-making problem, in which a contractor has to decide which of the three companies will be awarded a contract to manufacture a missile weapon system, and is a representative of a class of procurement judgment applications. Distinguishing features of our approach are as follows.

1) No preprocessing of the subcriteria scores (e.g., by ranking) is done, and therefore, no information is lost.
2) A wide range of mixed data can be used, from numbers to words. By not having to convert words into a preprocessed rank, information is again not lost.
3) Uncertainties about the subcriteria scores as well as their weights flow through all NWA calculations so that our final company performance FOUs not only contain ranking and similarity information but uncertainty information. No other existing method contains such uncertainty information.
4) Normalization automatically occurs in an NWA.

Although we have explained how the Per-C can be applied to a hierarchical multicriteria decision-making problem in the context of a specific procurement application, the methodology of this Per-C is quite general, and it can be applied to similar procurement applications. In addition, note that the Per-C is not limited to such applications only [52], [72]. It has been used to assist people in hierarchical and distributed decision making [51], [52], [72], social judgments [52], [72], etc.

Appendix
Computing the Novel Weighted Averages

Without loss of generality, in this appendix, it is assumed that: for the IWA, *all* subcriteria and weights are modeled as intervals; for the FWA, *all* subcriteria and weights are modeled as T1 FSs; and for the LWA, *all* subcriteria and weights are modeled as IT2 FSs. All other IWAs, FWAs, and LWAs shown in Fig. 5 can be viewed as special cases of these three cases.

A. Interval Weighted Average

The IWA is defined as

$$Y_{\mathrm{IWA}} \equiv \frac{\sum_{i=1}^{n} X_i W_i}{\sum_{i=1}^{n} W_i} = [l, r] \tag{A1}$$

where

$$X_i = [a_i,\ b_i], \quad i = 1, \ldots, n \tag{A2}$$

$$W_i = [c_i,\ d_i], \quad i = 1, \ldots, n \tag{A3}$$

and Y_{IWA} is also an interval completely determined by its two endpoints l and r, with

$$l = \min_{\substack{x_i \in X_i \\ w_i \in W_i}} \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i} = \min_{w_i \in W_i} \frac{\sum_{i=1}^{n} a_i w_i}{\sum_{i=1}^{n} w_i} \tag{A4}$$

$$r = \max_{\substack{x_i \in X_i \\ w_i \in W_i}} \frac{\sum_{i=1}^{n} x_i w_i}{\sum_{i=1}^{n} w_i} = \max_{w_i \in W_i} \frac{\sum_{i=1}^{n} b_i w_i}{\sum_{i=1}^{n} w_i}. \tag{A5}$$

The variables l and r can easily be computed by the Karnik–Mendel (KM) or enhanced Karnik–Mendel (EKM) algorithms [29], [43], [77].

Example 6: Suppose for $n = 5$, $\{x_i\}|_{i=1,\ldots,5} = \{9, 7, 5, 4, 1\}$ and $\{w_i\}|_{i=1,\ldots,5} = \{2, 1, 8, 4, 6\}$ so that the arithmetic WA $y_{\mathrm{AWA}} = 4.14$. Let λ denote any of these crisp numbers. In this example, for the IWA, $\lambda \to [\lambda - \delta, \lambda + \delta]$, where δ may be different for different λ, i.e.,

$$\{x_i\}|_{i=1,\ldots,5} \rightarrow \{[8.2, 9.8], [5.8, 8.2], [2.0, 8.0], [3.0, 5.0], [0.5, 1.5]\}$$
$$\{w_i\}|_{i=1,\ldots,5} \rightarrow \{[1.0, 3.0], [0.6, 1.4], [7.1, 8.9], [2.4, 5.6], [5.0, 7.0]\}.$$

It follows that $Y_{\mathrm{IWA}} = [2.02, 6.36]$. Note that the average of Y_{IWA} is 4.19, which is very close to the value of y_{AWA}. The important difference between y_{AWA} and Y_{IWA} is that the uncertainties about the subcriteria and weights have led to an uncertainty band for the IWA, and such a band may play a useful role in subsequent decision making.

B. Fuzzy Weighted Average

The FWA [16], [20], [21], [33], [35], [36] is defined as

$$Y_{\mathrm{FWA}} \equiv \frac{\sum_{i=1}^{n} X_i W_i}{\sum_{i=1}^{n} W_i} \tag{A6}$$

where X_i and W_i are T1 FSs, and Y_{FWA} is also a T1 FS. Note that (A6) is an *expressive* way to represent the FWA because it is not computed using multiplications, additions, and divisions, as expressed by it. Instead, it has been shown [36], [73] that the FWA can be computed by using the α-cut decomposition theorem [31], where each α-cut on Y_{FWA} is an IWA of the corresponding α-cuts on X_i and W_i, as described by the following algorithm.

1) For each $\alpha \in [0, 1]$, the corresponding α-cuts of the T1 FSs X_i and W_i are first computed, i.e., compute

$$X_i(\alpha) = [a_i(\alpha), b_i(\alpha)], \quad i = 1, \ldots, n \tag{A7}$$
$$W_i(\alpha) = [c_i(\alpha), d_i(\alpha)], \quad i = 1, \ldots, n. \tag{A8}$$

2) For each $\alpha \in [0, 1]$, compute the α-cut of the FWA by recognizing that it is an IWA, i.e., $Y_{\mathrm{FWA}}(\alpha) = Y_{\mathrm{IWA}}(\alpha)$, where

$$Y_{\mathrm{IWA}}(\alpha) = [l(\alpha), r(\alpha)] \tag{A9}$$

in which

$$l(\alpha) = \min_{\forall w_i(\alpha) \in [c_i(\alpha), d_i(\alpha)]} \frac{\sum_{i=1}^{n} a_i(\alpha) w_i(\alpha)}{\sum_{i=1}^{n} w_i(\alpha)} \tag{A10}$$

$$r(\alpha) = \max_{\forall w_i(\alpha) \in [c_i(\alpha), d_i(\alpha)]} \frac{\sum_{i=1}^{n} b_i(\alpha) w_i(\alpha)}{\sum_{i=1}^{n} w_i(\alpha)} \tag{A11}$$

and the KM or EKM algorithms [29], [43], [77] are used to compute $l(\alpha)$ and $r(\alpha)$.

3) Connect all left coordinates $(l(\alpha), \alpha)$ and all right coordinates $(r(\alpha), \alpha)$ to form the T1 FS Y_{FWA}.

Example 7: This is a continuation of Example 6 in which each interval is assigned a symmetric triangular T1 FS that is centered at the mid-point (λ) of the interval, has membership grade equal to one at that point, and is zero at the interval endpoints ($\lambda - \delta$ and $\lambda + \delta$) (see the triangle in Fig. 11). The FWA is depicted in Fig. 12. Although Y_{FWA} appears to be triangular, its sides are actually slightly curved.

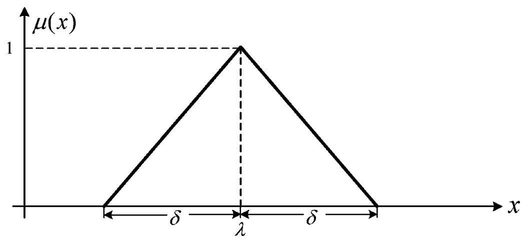

Fig. 11. T1 FS used in Example 7.

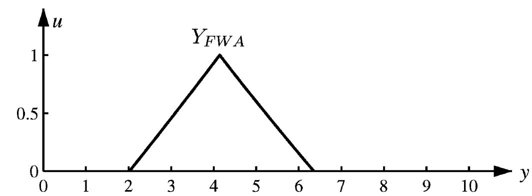

Fig. 12. FWA for Example 7.

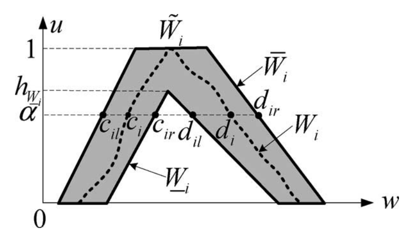

Fig. 13. $\tilde{W}_i$ and an α-cut. The dashed curve is an embedded T1 FS.

The support of Y_{FWA} is [2.02, 6.36], which is the same as Y_{IWA} (see Example 6). This will always occur because the support of Y_{FWA} is the $\alpha = 0$ α-cut, and this is Y_{IWA}.

The center of gravities of Y_{FWA} and Y_{IWA} are 4.15 and 4.19, respectively, and while close are not the same. The T1 FS Y_{FWA} indicates that more emphasis should be given to values of variable y that are closer to its apex, whereas the interval Y_{IWA} indicates that equal emphasis should be given to all values of variable y in its interval. The former reflects the propagation of the nonuniform uncertainties through the FWA and can be used in future decisions.

C. Linguistic Weighted Average

The LWA is defined as [73], [75]

$$\tilde{Y}_{\mathrm{LWA}} \equiv \frac{\sum_{i=1}^{n} \tilde{X}_i \tilde{W}_i}{\sum_{i=1}^{n} \tilde{W}_i} \tag{A12}$$

where X_i and W_i are IT2 FSs, and $\tilde{Y}_{\mathrm{FWA}}$ is also an IT2 FS. Again, (A12) is an *expressive* way to describe the LWA. To compute $\tilde{Y}_{\mathrm{LWA}}$, one only needs to compute its LMF $\underline{Y}_{\mathrm{LWA}}$ and UMF $\overline{Y}_{\mathrm{LWA}}$.

Let W_i be an embedded T1 FS [43] of $\tilde{W}_i$, as shown in Fig. 13. Because in (A12), $\tilde{X}_i$ only appears in the numerator of $\tilde{Y}_{\mathrm{LWA}}$,

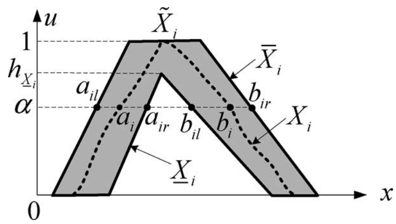

Fig. 14. $\tilde{X}_i$ and an α-cut.

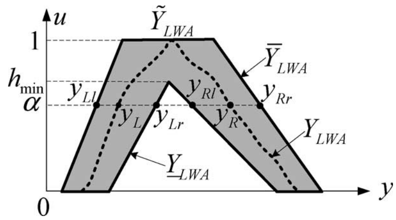

Fig. 15. $\tilde{Y}_{\mathrm{LWA}}$ and associated quantities.

it follows that

$$\underline{Y}_{\mathrm{LWA}} = \min_{\forall W_i \in [\underline{W}_i, \overline{W}_i]} \frac{\sum_{i=1}^n \underline{X}_i W_i}{\sum_{i=1}^n W_i} \tag{A13}$$

$$\overline{Y}_{\mathrm{LWA}} = \max_{\forall W_i \in [\underline{W}_i, \overline{W}_i]} \frac{\sum_{i=1}^n \overline{X}_i W_i}{\sum_{i=1}^n W_i}. \tag{A14}$$

The α-cut-based approach [73], [75] is also used to compute $\underline{Y}_{\mathrm{LWA}}$ and $\overline{Y}_{\mathrm{LWA}}$. First, the heights of $\underline{Y}_{\mathrm{LWA}}$ and $\overline{Y}_{\mathrm{LWA}}$ need to be determined. Because all UMFs are normal T1 FSs, $h_{\overline{Y}_{\mathrm{LWA}}} = 1$. Let $h_{\underline{X}_i}$ denote the height of $\underline{X}_i$ and $h_{\underline{W}_i}$ the height of $\underline{W}_i$. Let

$$h_{\min} = \min\{\min_{\forall i} h_{\underline{X}_i}, \min_{\forall i} h_{\underline{W}_i}\}. \tag{A15}$$

Then [75], $h_{\underline{Y}_{\mathrm{LWA}}} = h_{\min}$.

Let $[a_{il}(\alpha), b_{ir}(\alpha)]$ be an α-cut on $\overline{X}_i$, $[a_{ir}(\alpha), b_{il}(\alpha)]$ be an α-cut on $\underline{X}_i$ (see Fig. 14), $[c_{il}(\alpha), d_{ir}(\alpha)]$ be an α-cut on $\overline{W}_i$, $[c_{ir}(\alpha), d_{il}(\alpha)]$ be an α-cut on $\underline{W}_i$ (see Fig. 13), $[y_{Ll}(\alpha), y_{Rr}(\alpha)]$ be an α-cut on $\overline{Y}_{\mathrm{LWA}}$, and $[y_{Lr}(\alpha), y_{Rl}(\alpha)]$ be an α-cut on $\underline{Y}_{\mathrm{LWA}}$ (see Fig. 15), where the subscripts l and L mean *left*, and r and R mean *right*. The endpoints of the α-cuts on $\tilde{Y}_{\mathrm{LWA}}$ are computed as solutions to the following four optimization problems [73], [75]:

$$y_{Ll}(\alpha) = \min_{\forall w_i \in [c_{il}(\alpha), d_{ir}(\alpha)]} \frac{\sum_{i=1}^n a_{il}(\alpha) w_i}{\sum_{i=1}^n w_i}, \quad \alpha \in [0, 1] \tag{A16}$$

$$y_{Rr}(\alpha) = \max_{\forall w_i \in [c_{il}(\alpha), d_{ir}(\alpha)]} \frac{\sum_{i=1}^n b_{ir}(\alpha) w_i}{\sum_{i=1}^n w_i}, \quad \alpha \in [0, 1] \tag{A17}$$

$$y_{Lr}(\alpha) = \min_{\forall w_i \in [c_{ir}(\alpha), d_{il}(\alpha)]} \frac{\sum_{i=1}^n a_{ir}(\alpha) w_i}{\sum_{i=1}^n w_i}, \quad \alpha \in [0, h_{\min}] \tag{A18}$$

$$y_{Rl}(\alpha) = \max_{\forall w_i \in [c_{ir}(\alpha), d_{il}(\alpha)]} \frac{\sum_{i=1}^n b_{il}(\alpha) w_i}{\sum_{i=1}^n w_i}, \quad \alpha \in [0, h_{\min}]. \tag{A19}$$

Equations (A16)–(A19) are again computed by the KM or EKM algorithms [29], [43], [77].

From (A16), (A17), and Figs. 13 and 14, we observed that $y_{Ll}(\alpha)$ and $y_{Rr}(\alpha)$ only depend on the UMFs of $\tilde{X}_i$ and $\tilde{W}_i$, i.e., they are only computed from the corresponding α-cuts on the UMFs of $\tilde{X}_i$ and $\tilde{W}_i$; therefore

$$\overline{Y}_{\mathrm{LWA}} = \frac{\sum_{i=1}^n \overline{X}_i \overline{W}_i}{\sum_{i=1}^n \overline{W}_i}. \tag{A20}$$

Because all $\overline{X}_i$ and $\overline{W}_i$ are normal T1 FSs, $\overline{Y}_{\mathrm{LWA}}$ is also normal. The algorithm for computing $\overline{Y}_{\mathrm{LWA}}$ is as follows.

1) Select appropriate m α-cuts for $\overline{Y}_{\mathrm{LWA}}$ (e.g., divide [0, 1] into $m-1$ intervals, and set $\alpha_j = (j-1)/(m-1)$, $j = 1, 2, \ldots, m$).
2) For each α_j, find the corresponding α-cuts $[a_{il}(\alpha_j), b_{ir}(\alpha_j)]$ and $[c_{il}(\alpha_j), d_{ir}(\alpha_j)]$ on $\overline{X}_i$ and $\overline{W}_i$ $(i = 1, \ldots, n)$. Use a KM or EKM algorithm to find $y_{Ll}(\alpha_j)$ in (A16) and $y_{Rr}(\alpha_j)$ in (A17).
3) Connect all left coordinates $(y_{Ll}(\alpha_j), \alpha_j)$ and all right coordinates $(y_{Rr}(\alpha_j), \alpha_j)$ to form the T1 FS $\overline{Y}_{\mathrm{LWA}}$.

Similarly, we observed from (A18), (A19), and Figs. 13 and 14 that $y_{Lr}(\alpha)$ and $y_{Rl}(\alpha)$ only depend on the LMFs of $\tilde{X}_i$ and $\tilde{W}_i$; hence

$$\underline{Y}_{\mathrm{LWA}} = \frac{\sum_{i=1}^n \underline{X}_i \underline{W}_i}{\sum_{i=1}^n \underline{W}_i}. \tag{A21}$$

Unlike $\overline{Y}_{\mathrm{LWA}}$, which is a normal T1 FS, the height of $\underline{Y}_{\mathrm{LWA}}$ is $h_{\min}$, i.e., the minimum height of all $\underline{X}_i$ and $\underline{W}_i$. The algorithm for computing $\underline{Y}_{\mathrm{LWA}}$ is as follows.

1) Determine $h_{\underline{X}_i}$ and $h_{\underline{W}_i}$, $i = 1, \ldots, n$, and $h_{\min}$ in (A15).
2) Select appropriate p α-cuts for $\underline{Y}_{\mathrm{LWA}}$ (e.g., divide $[0, h_{\min}]$ into $p-1$ intervals and set $\alpha_j = h_{\min}(j-1)/(p-1)$, $j = 1, 2, \ldots, p$).
3) For each α_j, find the corresponding α-cuts $[a_{ir}(\alpha_j), b_{il}(\alpha_j)]$ and $[c_{ir}(\alpha_j), d_{il}(\alpha_j)]$ on $\underline{X}_i$ and $\underline{W}_i$. Use a KM or EKM algorithm to find $y_{Lr}(\alpha_j)$ in (A18) and $y_{Rl}(\alpha_j)$ in (A19).
4) Connect all left coordinates $(y_{Lr}(\alpha_j), \alpha_j)$ and all right coordinates $(y_{Rl}(\alpha_j), \alpha_j)$ to form the T1 FS $\underline{Y}_{\mathrm{LWA}}$.

In summary, computing $\tilde{Y}_{\mathrm{LWA}}$ is equivalent to computing two FWAs: $\overline{Y}_{\mathrm{LWA}}$ and $\underline{Y}_{\mathrm{LWA}}$. A flowchart for computing $\underline{Y}_{\mathrm{LWA}}$ and $\overline{Y}_{\mathrm{LWA}}$ is given in Fig. 16. For triangular or trapezoidal IT2 FSs, it is possible to reduce the number of α-cuts for both $\underline{Y}_{\mathrm{LWA}}$ and $\overline{Y}_{\mathrm{LWA}}$ by choosing them only at *turning points*, i.e., points on the LMFs and UMFs of $\tilde{X}_i$ and $\tilde{W}_i$ $(i = 1, 2, \ldots, n)$ at which the slope of these functions changes.

In order to communicate effectively about the LWA, some measures of the uncertainty that is associated with $\tilde{Y}_{\mathrm{LWA}}$ are needed [74]. The centroid [29] of the LWA is one such measure.

Definition 5: Let y_i $(i = 1, 2, \ldots, N)$ be the discretizations in the domain of the primary variable of $\tilde{Y}_{\mathrm{LWA}}$. Then, the

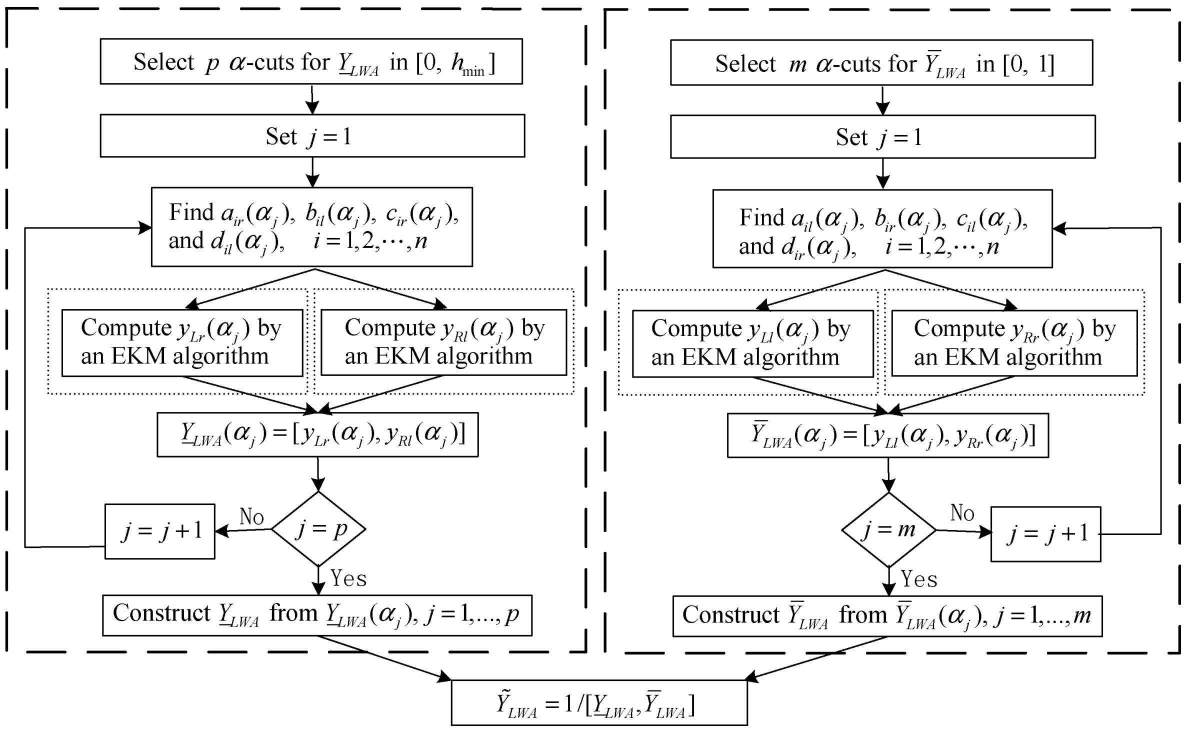

Fig. 16. Flowchart for computing the LWA [75].

membership of y_i, i.e., $\mu_{\tilde{Y}_{\rm LWA}}(y_i)$, is an interval, i.e.,

$$\mu_{\tilde{Y}_{\rm LWA}}(y_i) = [\underline{Y}_{\rm LWA}(y_i), \overline{Y}_{\rm LWA}(y_i)] \tag{A22}$$

and the *centroid* of $\tilde{Y}_{\rm LWA}$, which is a special IWA, is defined as

$$C_{\tilde{Y}_{\rm LWA}} = \frac{\sum_{i=1}^N y_i \mu_{\tilde{Y}_{\rm LWA}}(y_i)}{\sum_{i=1}^N \mu_{\tilde{Y}_{\rm LWA}}(y_i)} \equiv [c_l, c_r] \tag{A23}$$

where

$$c_l = \min_{\forall \mu(y_i)\in[\underline{Y}_{\rm LWA}(y_i),\overline{Y}_{\rm LWA}(y_i)]} \frac{\sum_{i=1}^N y_i\mu(y_i)}{\sum_{i=1}^N \mu(y_i)} \tag{A24}$$

$$c_r = \max_{\forall \mu(y_i)\in[\underline{Y}_{\rm LWA}(y_i),\overline{Y}_{\rm LWA}(y_i)]} \frac{\sum_{i=1}^N y_i\mu(y_i)}{\sum_{i=1}^N \mu(y_i)}. \tag{A25}$$

The variables c_l and c_r are also computed by the KM or EKM algorithms.

Definition 6: The *average centroid* (center of centroid) of $\tilde{Y}_{\rm LWA}$ is defined as

$$c_{\tilde{Y}_{\rm LWA}} = (c_l + c_r)/2. \tag{A26}$$

The *half-length* of $\tilde{Y}_{\rm LWA}$ is defined as

$$\delta_{\tilde{Y}_{\rm LWA}} = (c_r - c_l)/2. \tag{A27}$$

Example 8: This is a continuation of Example 7, where each subcriterion and weight is now assigned an FOU, i.e., for a 50% symmetrical blurring of the T1 MF depicted in Fig. 11 (see Fig. 17). The left half of each FOU has support on the x (w)-axis given by the interval of real numbers $[(\lambda-\delta)-0.5\delta, (\lambda-\delta)+0.5\delta]$, and the right-half FOU has support on the x-axis given by the interval of real numbers $[(\lambda+\delta)-0.5\delta, (\lambda+\delta)+0.5\delta]$. The UMF is a triangle defined by the three points $(\lambda-\delta-0.5\delta, 0), (\lambda, 1), (\lambda+\delta+0.5\delta, 0)$, and the LMF is a triangle defined by the three points $(\lambda-\delta+0.5\delta, 0), (\lambda, 1), (\lambda+\delta-0.5\delta, 0)$. The resulting subcriterion and weight FOUs are depicted in Fig. 18(a) and (b), respectively, and $\tilde{Y}_{\rm LWA}$ is depicted in Fig. 18(c). Although $\tilde{Y}_{\rm LWA}$ appears to be symmetrical, it is not. The support of the left-hand side of $\tilde{Y}_{\rm LWA}$ is [0.85, 3.10], and the support of the right-hand side of $\tilde{Y}_{\rm LWA}$ is [5.22, 7.56]; hence, the length of the support of the left-hand side of $\tilde{Y}_{\rm LWA}$ is 2.25, whereas the length of the support of the right-hand side of $\tilde{Y}_{\rm LWA}$ is 2.34. In addition, $C_{\tilde{Y}_{\rm LWA}} = [3.38, 4.96]$, and $c_{\tilde{Y}_{\rm LWA}} = 4.17$.

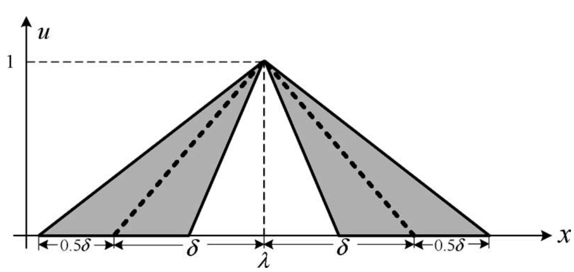

Fig. 17. IT2 FS used in Example 8. The dashed lines are the corresponding T1 FS used in Example 7.

Comparing Figs. 12 and 18(c), we observed that $\tilde{Y}_{\rm LWA}$ is spread out over a larger range of values than is $Y_{\rm FWA}$, thus reflecting the additional uncertainties in the LWA due to the

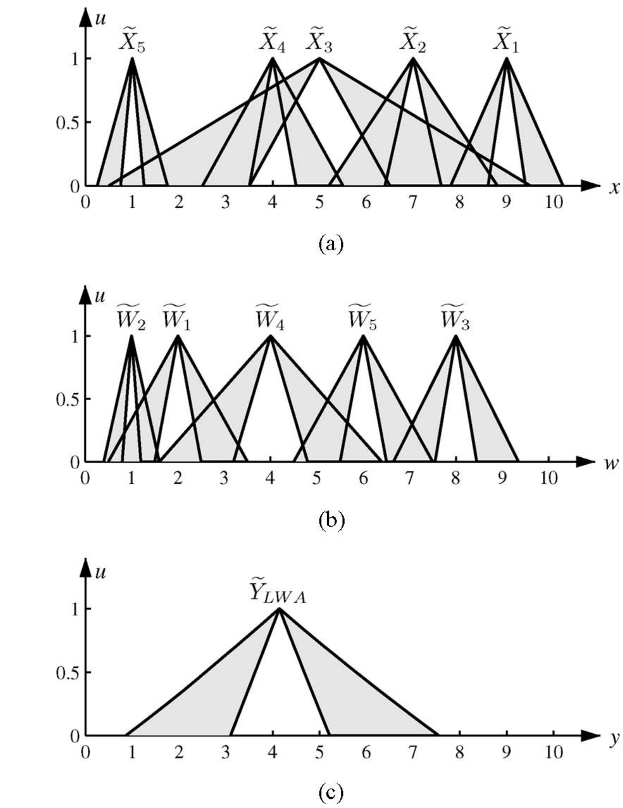

Fig. 18. Example 8. (a) $\tilde{X}_i$, (b) $\widetilde{W}_i$, and (c) $\tilde{Y}_{\mathrm{LWA}}$.

blurring of subcriteria and weights. This information can be used in future decisions.

Another way to interpret $\tilde{Y}_{\mathrm{LWA}}$ is to associate values of y that have the largest vertical intervals (i.e., primary memberships) with values of greatest uncertainty; hence, there is no uncertainty at the three vertices of the UMF, and, e.g., for the right half of $\tilde{Y}_{\mathrm{LWA}}$, uncertainty increases from the apex of the UMF, reaching its largest value at the right vertex of the LMF and then decreases to zero at the right vertex of the UMF.

ACKNOWLEDGMENT

The authors would like to thank the reviewers for their insightful comments that have helped improve this paper significantly.

REFERENCES

[1] Tactical ballistic missile. (2010, Mar. 8). [Online]. Available: http://en.wikipedia.org/wiki/Tactical_ballistic_missile

[2] *Decision Making* (ser. Harvard Business Essentials). Boston, MA: Harvard Business School Press, 2005.

[3] M. C. R. Ailles, "Antiballistic missile system evaluation," *Army Res. Dev. News Mag.*, pp. 20–21, May/Jun. 1974.

[4] P. Baguley, T. Page, V. Koliza, and P. Maropoulos, "Time to market prediction using type-2 fuzzy sets," *J. Manuf. Technol. Manage.*, vol. 17, no. 4, pp. 513–520, 2006.

[5] G. Beliakov, A. Pradera, and T. Calvo, *Aggregation Functions: A Guide for Practitioners*. Berlin/Heidelberg, Germany: Springer-Verlag, 2007.

[6] J. Buckley and T. Feuring, "Computing with words in control," in *Computing With Words in Information/Intelligent Systems 2: Applications*, L. A. Zadeh and J. Kacprzyk, Eds. Heidelberg, Germany: Physica-Verlag, 1999, pp. 289–304.

[7] T. Calvo, G. Mayor, and R. Mesiar, Eds., *Aggregation Operators: New Trends and Applications*. Heidelberg, Germany: Physica-Verlag, 2002.

[8] O. Castillo, L. T. Aguilar, N. R. Cazarez-Castro, and S. Cardenas, "Systematic design of a stable type-2 fuzzy logic controller," *Appl. Soft Comput.*, vol. 8, no. 3, pp. 1274–1279, 2008.

[9] P.-T. Chang and K.-C. Hung, "Applying the fuzzy-weighted-average approach to evaluate network security systems," *Comput. Math. Appl.*, vol. 49, pp. 1797–1814, 2005.

[10] S.-M. Chen, "Evaluating weapon systems using fuzzy arithmetic operations," *Fuzzy Sets Syst.*, vol. 77, pp. 265–276, 1996.

[11] S.-M. Chen, "A new method for evaluating weapon systems using fuzzy set theory," *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans*, vol. 26, pp. 493–497, Jul. 1996.

[12] C.-H. Cheng, "Evaluating naval tactical missile systems by fuzzy AHP based on the grade value of membership function," *Eur. J. Oper. Res.*, vol. 96, pp. 343–350, 1996.

[13] C.-H. Cheng, "Evaluating weapon systems using ranking fuzzy numbers," *Fuzzy Sets Syst.*, vol. 107, pp. 25–35, 1999.

[14] G. Choquet, "Theory of capacities," *Ann. Inst. Fourier*, vol. 5, pp. 131–295, 1953.

[15] B. K. Das, "Test and evaluation of tactical missile system using electro-optical tracking system," *Int. Test Eval. Assoc. J.*, vol. 30, pp. 143–148, 2009.

[16] W. M. Dong and F. S. Wong, "Fuzzy weighted averages and implementation of the extension principle," *Fuzzy Sets Syst.*, vol. 21, pp. 183–199, 1987.

[17] D. Dubois and H. Prade, "Criteria aggregation and ranking of alternatives in the framework of fuzzy set theory," in *Fuzzy Sets Decis. Anal.*, H. Zimmermann, L. Zadeh, and B. Gaines, Eds. Amsterdam, The Netherlands: North-Holland, Elsevier, 1984, pp. 209–240.

[18] E. J. Eichblatt, *Test and Evaluation of the Tactical Missile*. Washington, DC: AIAA, 1989.

[19] L. Gu and Y. Q. Zhang, "Web shopping expert using new interval type-2 fuzzy reasoning," *Soft Comput.*, vol. 11, no. 8, pp. 741–751, 2007.

[20] Y.-Y. Guh, C.-C. Hon, and E. S. Lee, "Fuzzy weighted average: The linear programming approach via Charnes and Cooper's rule," *Fuzzy Sets Syst.*, vol. 117, pp. 157–160, 2001.

[21] Y.-Y. Guh, C.-C. Hon, K.-M. Wang, and E. S. Lee, "Fuzzy weighted average: A max-min paired elimination method," *J. Comput. Math. Appl.*, vol. 32, pp. 115–123, 1996.

[22] H. Hagras, "A hierarchical type-2 fuzzy logic control architecture for autonomous mobile robots," *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 4, pp. 524–539, Aug. 2004.

[23] H. Hagras, "Type-2 FLCs: A new generation of fuzzy controllers," *IEEE Comput. Intell. Mag.*, vol. 2, no. 1, pp. 30–43, Feb. 2007.

[24] F. Herrera, "A sequential selection process in group decision making with linguistic assessment," *Inf. Sci.*, vol. 85, pp. 223–239, 1995.

[25] F. Herrera and E. Herrera-Viedma, "Aggregation operators for linguistic weighted information," *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans*, vol. 27, no. 5, pp. 646–656, Sep. 1997.

[26] F. Herrera and L. Martinez, "A 2-tuple fuzzy linguistic representation model for computing with words," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 6, pp. 746–752, Dec. 2000.

[27] J. Kacprzyk and S. Zadrożny, "Computing with words in intelligent database querying: Standalone and internet-based applications," *Inf. Sci.*, vol. 34, pp. 71–109, 2001.

[28] C. Kao and S.-T. Liu, "Competitiveness of manufacturing firms: An application of fuzzy weighted average," *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans*, vol. 29, no. 6, pp. 661–667, Sep. 1999.

[29] N. N. Karnik and J. M. Mendel, "Centroid of a type-2 fuzzy set," *Inf. Sci.*, vol. 132, pp. 195–220, 2001.

[30] C. S. Kim, D. S. Kim, and J. S. Park, "A new fuzzy resolution principle based on the antonym," *Fuzzy Sets Syst.*, vol. 113, pp. 299–307, 2000.

[31] G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Upper Saddle River, NJ: Prentice-Hall, 1995.

[32] J. Lawry, "A methodology for computing with words," *Int. J. Approx. Reason.*, vol. 28, pp. 51–89, 2001.

[33] D. H. Lee and M. H. Kim, "Database summarization using fuzzy ISA hierarchies," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 27, no. 4, pp. 68–78, Aug. 1997.

[34] C.-T. Lin and C.-T. Chen, "A fuzzy-logic-based approach for new product Go/NoGo decision at the front end," *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans*, vol. 34, no. 1, pp. 132–142, Jan. 2004.

[35] T.-S. Liou and M.-J. J. Wang, "Fuzzy weighted average: An improved algorithm," *Fuzzy Sets Syst.*, vol. 49, pp. 307–315, 1992.

[36] F. Liu and J. M. Mendel, "Aggregation using the fuzzy weighted average, as computed using the Karnik-Mendel Algorithms," *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 1, pp. 1–12, Feb. 2008.

[37] F. Liu and J. M. Mendel, "Encoding words into interval type-2 fuzzy sets using an interval approach," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 6, pp. 1503–1521, Aug. 2008.

[38] J. Lu, G. Zhang, D. Ruan, and F. Wu, *Multi-Objective Group Decision Making*. London, U.K.: Imperial College Press, 2007.

[39] C. Lynch, H. Hagras, and V. Callaghan, "Embedded type-2 FLC for real-time speed control of marine and traction diesel engines," in *Proc. FUZZ IEEE*, Reno, NV, May 2005, pp. 347–352.

[40] M. Margaliot and G. Langholz, "Fuzzy control of a benchmark problem: A computing with words approach," *IEEE Trans. Fuzzy Syst.*, vol. 12, no. 2, pp. 230–235, Apr. 2004.

[41] R. Martinez-Marroquin, O. Castillo, and L. T. Aguilar, "Optimization of interval type-2 fuzzy logic controllers for a perturbed autonomous wheeled mobile robot using genetic algorithms," *Inf. Sci.*, vol. 179, no. 13, pp. 2158–2174, 2009.

[42] P. Melin, J. Urias, D. Solano, M. Soto, M. Lopez, and O. Castillo, "Voice recognition with neural networks, type-2 fuzzy logic and genetic algorithms," *J. Eng. Lett.*, vol. 13, no. 2, pp. 108–116, 2006.

[43] J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. Upper Saddle River, NJ: Prentice-Hall, 2001.

[44] J. M. Mendel, "An architecture for making judgments using computing with words," *Int. J. Appl. Math. Comput. Sci.*, vol. 12, no. 3, pp. 325–335, 2002.

[45] J. M. Mendel, "Computing with words and its relationships with fuzzistics," *Inf. Sci.*, vol. 177, pp. 988–1006, 2007.

[46] J. M. Mendel, "Computing with words: Zadeh, Turing, Popper and Occam," *IEEE Comput. Intell. Mag.*, vol. 2, no. 4, pp. 10–17, Nov. 2007.

[47] J. M. Mendel, H. Hagras, and R. I. John. (2010, Mar. 8). Standard background material about interval type-2 fuzzy logic systems that can be used by all authors [Online]. Available: http://ieee-cis.org/_files/standards.t2.win.pdf

[48] J. M. Mendel and R. I. John, "Type-2 fuzzy sets made simple," *IEEE Trans. Fuzzy Syst.*, vol. 10, no. 2, pp. 117–127, Apr. 2002.

[49] J. M. Mendel, R. I. John, and F. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 6, pp. 808–821, Dec. 2006.

[50] J. M. Mendel and D. Wu, "Perceptual reasoning for perceptual computing," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 6, pp. 1550–1564, Dec. 2008.

[51] J. M. Mendel and D. Wu, "Computing with words for hierarchical distributed decision making," in *Computational Intelligence in Complex Decision Systems*, D. Ruan, Ed. Paris, France: Atlantis, 2009.

[52] J. M. Mendel and D. Wu, *Perceptual Computing: Aiding People in Making Subjective Judgments*, Hoboken, NJ: Wiley–IEEE Press, Jun. 2010.

[53] H. B. Mitchell, "Pattern recognition using type II fuzzy sets," *Inf. Sci.*, vol. 170, no. 2–4, pp. 409–418, 2005.

[54] D.-L. Mon, C.-H. Cheng, and J.-L. Lin, "Evaluating weapon system using fuzzy analytic hierarchy process based on entropy weight," *Fuzzy Sets Syst.*, vol. 62, pp. 127–134, 1994.

[55] T. Murofushi and M. Sugeno, "An interpretation of fuzzy measure and the Choquet integral as an integral with respect to a fuzzy measure," *Fuzzy Sets Syst.*, vol. 29, pp. 201–227, 1989.

[56] A. Niewiadomski, *On Two Possible Roles of Type-2 Fuzzy Sets in Linguistic Summaries* (Lecture Notes Comput. Sci., 3528). Berlin, Germany: Springer-Verlag, 2005, pp. 341–347.

[57] V. Novák, "Antonyms and linguistic quantifiers in fuzzy logic," *Fuzzy Sets Syst.*, vol. 124, pp. 335–351, 2001.

[58] S. K. Pal, L. Polkowski, and A. Skowron, Eds., *Rough-Neural Computing: Techniques for Computing with Words*. Heidelberg, Germany: Springer-Verlag, 2003.

[59] S. H. Rubin, "Computing with words," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 29, no. 4, pp. 518–524, Aug. 1999.

[60] R. Sepulveda, O. Castillo, P. Melin, A. R. Diaz, and O. Montiel, "Experimental study of intelligent controllers under uncertainty using type-1 and type-2 fuzzy logic," *Inf. Sci.*, vol. 177, no. 10, pp. 2023–2048, 2007.

[61] A. D. Soto and E. Trillas, "On antonym and negate in fuzzy logic," *Int. J. Intell. Syst.*, vol. 14, pp. 295–303, 1999.

[62] E. Triantaphyllou, *Multi-Criteria Decision Making Methods: A Comparative Study*. Norwell, MA: Kluwer, 2000.

[63] I. B. Türkşen, "Type 2 representation and reasoning for CWW," *Fuzzy Sets Syst.*, vol. 127, pp. 17–36, 2002.

[64] G.-H. Tzeng and J.-Y. Teng, "Transportation investment project selection with fuzzy multiobjectives," *Transport. Planning Technol.*, vol. 17, pp. 91–112, 1993.

[65] L. V. Vanegas and A. W. Labib, "Application of new fuzzy-weighted average (NFWA) method to engineering design evaluation," *Int. J. Prod. Res.*, vol. 39, no. 6, pp. 1147–1162, 2001.

[66] I. Vlachos and G. Sergiadis, "Subsethood, entropy, and cardinality for interval-valued fuzzy sets—An algebraic derivation," *Fuzzy Sets Syst.*, vol. 158, pp. 1384–1396, 2007.

[67] T. S. Wallsten and D. V. Budescu, "A review of human linguistic probability processing: General principles and empirical evidence," *Knowl. Eng. Rev.*, vol. 10, no. 1, pp. 43–62, 1995.

[68] H. Wang and D. Qiu, "Computing with words via Turing machines: A formal approach," *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 6, pp. 742–753, Dec. 2003.

[69] J.-H. Wang and J. Hao, "A new version of 2-tuple fuzzy linguistic representation model for computing with words," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 3, pp. 435–445, Jun. 2006.

[70] M.-J. J. Wang and T.-C. Chang, "Tool steel materials selection under fuzzy environment," *Fuzzy Sets Syst.*, vol. 72, no. 3, pp. 263–270, 1995.

[71] P. Wang, Ed., *Computing With Words*. New York: Wiley, 2001.

[72] D. Wu, "Intelligent systems for decision support," Ph.D. dissertation, Univ. Southern Calif., Los Angeles, CA, May 2009.

[73] D. Wu and J. M. Mendel, "Aggregation using the linguistic weighted average and interval type-2 fuzzy sets," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 6, pp. 1145–1161, Dec. 2007.

[74] D. Wu and J. M. Mendel, "Uncertainty measures for interval type-2 fuzzy sets," *Inf. Sci.*, vol. 177, no. 23, pp. 5378–5393, 2007.

[75] D. Wu and J. M. Mendel, "Corrections to 'Aggregation using the linguistic weighted average and interval type-2 fuzzy sets'," *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 6, pp. 1664–1666, Sep. 2008.

[76] D. Wu and J. M. Mendel, "A comparative study of ranking methods, similarity measures and uncertainty measures for interval type-2 fuzzy sets," *Inf. Sci.*, vol. 179, no. 8, pp. 1169–1192, 2009.

[77] D. Wu and J. M. Mendel, "Enhanced Karnik-Mendel algorithms," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 923–934, Jul. 2009.

[78] D. Wu and J. M. Mendel, "Interval type-2 fuzzy set subsethood measures as a decoder for perceptual reasoning," Signal Image Process. Inst., Ming Hsieh Dept. Electr. Eng., Univ. Southern Calif., Los Angeles, USC-SIPI Rep. 398, 2009.

[79] D. Wu and J. M. Mendel, "Perceptual reasoning for perceptual computing: A similarity-based approach," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 6, pp. 1397–1411, Dec. 2009.

[80] D. Wu and W. W. Tan, "Genetic learning and performance evaluation of type-2 fuzzy logic controllers," *Int. J. Eng. Appl. Artif. Intell.*, vol. 19, no. 8, pp. 829–841, 2006.

[81] D. Wu and W. W. Tan, "A simplified type-2 fuzzy controller for real-time control," *ISA Trans.*, vol. 15, no. 4, pp. 503–516, 2006.

[82] R. Yager, "Fuzzy subsets of type-2 in decisions," *J. Cybern.*, vol. 10, pp. 137–159, 1980.

[83] R. Yager, "On ordered weighted averaging aggregation operators in multi-criteria decision making," *IEEE Trans. Syst., Man, Cybern.*, vol. 18, pp. 183–190, Jan./Feb. 1988.

[84] R. Yager, "Aproximate reasoning as a basis for computing with words," in *Computing With Words in Information/Intelligent Systems 1: Foundations*. L. A. Zadeh and J. Kacprzyk, Eds. Heidelberg, Germany: Physica-Verlag, 1999, pp. 50–77.

[85] R. Yager, "On the retranslation process in Zadeh's paradigm of computing with words," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 34, no. 2, pp. 1184–1195, Apr. 2004.

[86] M. Ying, "A formal model of computing with words," *IEEE Trans. Fuzzy Syst.*, vol. 10, no. 5, pp. 640–652, Oct. 2002.

[87] L. A. Zadeh, "Fuzzy sets," *Inf. Control*, vol. 8, pp. 338–353, 1965.

[88] L. A. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-1," *Inf. Sci.*, vol. 8, pp. 199–249, 1975.

[89] L. A. Zadeh, "Fuzzy logic = Computing with words," *IEEE Trans. Fuzzy Syst.*, vol. 4, pp. 103–111, May 1996.

[90] L. A. Zadeh, "From computing with numbers to computing with words—From manipulation of measurements to manipulation of perceptions," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 46, no. 1, pp. 105–119, Jul. 1999.

[91] L. A. Zadeh, "Toward a generalized theory of uncertainty (GTU)—An outline," *Inf. Sci.*, vol. 172, pp. 1–40, 2005.

[92] L. A. Zadeh, "Toward human level machine intelligence—Is it achievable? The need for a paradigm shift," *IEEE Comput. Intell. Mag.*, vol. 3, no. 3, pp. 11–22, Aug. 2008.

[93] L. A. Zadeh and J. Kacprzyk, Eds., *Computing With Words in Information/Intelligent Systems: 1. Foundations, 2. Applications*. Heidelberg, Germany: Physica-Verlag, 1999.

[94] P. Zarchan, *Tactical and Strategic Missile Guidance*. Washington, DC: AIAA, 1984.

[95] J. Zeng and Z.-Q. Liu, “Type-2 fuzzy hidden Markov models and their applications to speech recognition,” *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 3, pp. 454–467, Jun. 2006.

Dongrui Wu (S’05–M’09) received the B.E. degree in automatic control from the University of Science and Technology of China, Hefei, China, in 2003, the M.Eng. degree in electrical engineering from the National University of Singapore, Singapore, in 2005, and the Ph.D. degree in electrical engineering from the University of Southern California, Los Angeles, in 2009.

Currently, he is with the Institute for Creative Technologies and the Signal Analysis and Interpretation Laboratory, Viterbi School of Engineering, University of Southern California. His research interest includes affective computing, computational intelligence, intelligent control, machine learning, and speech and physiological signal processing.

Dr. Wu received the Best Student Paper Award from the IEEE International Conference on Fuzzy Systems, Reno, NV, in 2005.

Jerry M. Mendel (S’59–M’61–SM’72–F’78–LF’04) received the Ph.D. degree in electrical engineering from the Polytechnic Institute of Brooklyn, Brooklyn, NY.

Since 1974, he has been with the University of Southern California, Los Angeles, where he is currently a Professor of electrical engineering with the Signal and Image Processing Institute, Ming Hsieh Department of Electrical Engineering. He is the author or coauthor of more than 470 technical papers and is the author and/or editor of eight books, including *Uncertain Rule-based Fuzzy Logic Systems: Introduction and New Directions* (Englewood Cliffs, NJ: Prentice-Hall, 2001). His research interests include type-2 fuzzy logic systems and their applications to a wide range of problems, including smart oilfield technology and computing with words.

Prof. Mendel is a Distinguished Member of the IEEE Control Systems Society. He was the President of the IEEE Control Systems Society during 1986. He is the Chairman of the Fuzzy Systems Technical Committee and an elected member of the Administrative Committee of the IEEE Computational Intelligence Society. He was the recipient of the 1983 Best Transactions Paper Award of the IEEE Geoscience and Remote Sensing Society, the 1992 Signal Processing Society Paper Award, the 2002 TRANSACTIONS ON FUZZY SYSTEMS Outstanding Paper Award, the 1984 IEEE Centennial Medal, an IEEE Third Millenium Medal, a Pioneer Award from the IEEE Granular Computing Conference, May 2006, for Outstanding Contributions in Type-2 Fuzzy Systems, and the 2008 Fuzzy Systems Pioneer Award from the IEEE Computational Intelligence Society.